8. Seasonal ARIMA models

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1 Backshift notation reviewed

2 Seasonal ARIMA models

3 ARIMA vs ETS
A very useful notational device is the backward shift operator, $B$, which is used as follows:

$$By_t = y_{t-1}.$$ 

In other words, $B$, operating on $y_t$, has the effect of shifting the data back one period. Two applications of $B$ to $y_t$ shifts the data back two periods:

$$B(By_t) = B^2 y_t = y_{t-2}.$$ 

For monthly data, if we wish to shift attention to “the same month last year,” then $B^{12}$ is used, and the notation is $B^{12} y_t = y_{t-12}$. 

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Backshift notation

- **First difference:** $1 - B$.
- **Double difference:** $(1 - B)^2$.
- **$d$th-order difference:** $(1 - B)^d y_t$.
- **Seasonal difference:** $1 - B^m$.
- **Seasonal difference followed by a first difference:** $(1 - B)(1 - B^m)$.
- **Multiply terms together to see the combined effect:**

\[
(1 - B)(1 - B^m)y_t = (1 - B - B^m + B^{m+1})y_t
= y_t - y_{t-1} - y_{t-m} + y_{t-m-1}.
\]
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\]
Backshift notation for ARIMA

**ARMA model:**

\[ y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + e_t + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q} \]

\[ = c + \phi_1 By_t + \cdots + \phi_p B^p y_t + e_t + \theta_1 Be_t + \cdots + \theta_q B^q e_t \]

\[ \phi(B)y_t = c + \theta(B)e_t \]

where \( \phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p \)

and \( \theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q. \)

**ARIMA(1,1,1) model:**

\[ (1 - \phi_1 B) (1 - B)y_t = c + (1 + \theta_1 B)e_t \]
ARMA model:

\[ y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + e_t + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q} \]

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Backshift notation for ARIMA

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  \[ y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + e_t + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q} \]
  \[ = c + \phi_1 B y_t + \cdots + \phi_p B^p y_t + e_t + \theta_1 B e_t + \cdots + \theta_q B^q e_t \]
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- **ARIMA(1,1,1) model:**
  \[ (1 - \phi_1 B) (1 - B)y_t = c + (1 + \theta_1 B)e_t \]
  ↑
  First difference
Backshift notation for ARIMA

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  \[
  \uparrow
  \]
  AR(1)
ARMA model:
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= c + \phi_1 By_t + \cdots + \phi_p B^p y_t + e_t + \theta_1 Be_t + \cdots + \theta_q B^q e_t
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ARIMA(1,1,1) model:
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(1 - \phi_1 B) (1 - B) y_t = c + (1 + \theta_1 B) e_t
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MA(1)
Outline

1. Backshift notation reviewed

2. Seasonal ARIMA models

3. ARIMA vs ETS
Seasonal ARIMA models

ARIMA \((p, d, q) (P, D, Q)_m\)

where \(m = \text{number of periods per season.}\)
Seasonal ARIMA models

\[
\text{ARIMA } (p, d, q) \underbrace{(P, D, Q)}_{\text{Non-seasonal part of the model}} m
\]

where \( m = \text{number of periods per season.} \)
Seasonal ARIMA models

ARIMA $(p, d, q)$ $(P, D, Q)_m$

where $m = \text{number of periods per season.}$
Seasonal ARIMA models

E.g., ARIMA(1, 1, 1)(1, 1, 1)_4 model (without constant)

\[(1 - \phi_1 B)(1 - \Phi_1 B^4)(1 - B)(1 - B^4)y_t = (1 + \theta_1 B)(1 + \Theta_1 B^4)e_t.\]
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(Non-seasonal difference)
Seasonal ARIMA models

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\[
\begin{pmatrix}
\text{Non-seasonal} \\
\text{MA(1)}
\end{pmatrix}
\]
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All the factors can be multiplied out and the general model written as follows:

\[y_t = (1 + \phi_1)y_{t-1} - \phi_1 y_{t-2} + (1 + \Phi_1)y_{t-4} - (1 + \phi_1 + \Phi_1 + \phi_1 \Phi_1)y_{t-5} + (\phi_1 + \phi_1 \Phi_1)y_{t-6} - \phi_1 y_{t-8} + (\Phi_1 + \phi_1 \Phi_1)y_{t-9} - \phi_1 \Phi_1 y_{t-10} + e_t + \theta_1 e_{t-1} + \Theta_1 e_{t-4} + \theta_1 \Theta_1 e_{t-5}.\]
In the US Census Bureau uses the following models most often:

<table>
<thead>
<tr>
<th>Model</th>
<th>Transformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>ARIMA(0,1,1)(0,1,1)m</td>
<td>with log</td>
</tr>
<tr>
<td>ARIMA(0,1,2)(0,1,1)m</td>
<td>with log</td>
</tr>
<tr>
<td>ARIMA(2,1,0)(0,1,1)m</td>
<td>with log</td>
</tr>
<tr>
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<td>with log</td>
</tr>
<tr>
<td>ARIMA(2,1,2)(0,1,1)m</td>
<td>with no</td>
</tr>
</tbody>
</table>
The seasonal part of an AR or MA model will be seen in the seasonal lags of the PACF and ACF.

**ARIMA(0,0,0)(0,0,1)_12 will show:**
- a spike at lag 12 in the ACF but no other significant spikes.
- The PACF will show exponential decay in the seasonal lags; that is, at lags 12, 24, 36, ....

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**ARIMA(0,0,0)(1,0,0)_12 will show:**
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**ARIMA(0,0,0)(0,0,1)_{12} will show:**
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**ARIMA(0,0,0)(1,0,0)_{12} will show:**
- exponential decay in the seasonal lags of the ACF.
- a single significant spike at lag 12 in the PACF.
The seasonal part of an AR or MA model will be seen in the seasonal lags of the PACF and ACF.

**ARIMA(0,0,0)(0,0,1)_{12} will show:**

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**ARIMA(0,0,0)(1,0,0)_{12} will show:**

- exponential decay in the seasonal lags of the ACF
- a single significant spike at lag 12 in the PACF.
Seasonal ARIMA models

The seasonal part of an AR or MA model will be seen in the seasonal lags of the PACF and ACF.

**ARIMA(0,0,0)(0,0,1)\textsubscript{12} will show:**
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**ARIMA(0,0,0)(1,0,0)\textsubscript{12} will show:**
- exponential decay in the seasonal lags of the ACF
- a single significant spike at lag 12 in the PACF.
The seasonal part of an AR or MA model will be seen in the seasonal lags of the PACF and ACF.

**ARIMA(0,0,0)(0,0,1)_{12} will show:**
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**ARIMA(0,0,0)(1,0,0)_{12} will show:**
- exponential decay in the seasonal lags of the ACF
- a single significant spike at lag 12 in the PACF.
European quarterly retail trade

Retail index

Year

Forecasting: Principles and Practice
European quarterly retail trade

> plot(euretail)
European quarterly retail trade

Seasonal ARIMA models
European quarterly retail trade

\[ \text{tsdisplay}(\text{diff(euretail,4)}) \]

\[ \text{ACF} \]

\[ \text{PACF} \]
> tsdisplay(diff(diff(diff(euretail,4))))
- $d = 1$ and $D = 1$ seems necessary.
- Significant spike at lag 1 in ACF suggests non-seasonal MA(1) component.
- Significant spike at lag 4 in ACF suggests seasonal MA(1) component.
- Initial candidate model: ARIMA(0,1,1)(0,1,1)$_4$.
- We could also have started with ARIMA(1,1,0)(1,1,0)$_4$.

```r
fit <- Arima(euretail, order=c(0,1,1), seasonal=c(0,1,1))
```

```
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```
\[ d = 1 \text{ and } D = 1 \text{ seems necessary.} \]

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fit <- Arima(euretail, order=c(0,1,1), seasonal=c(0,1,1))
tsddisplay(residuals(fit))
```
European quarterly retail trade

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\[
\text{fit} \leftarrow \text{Arima(\text{euretail}, \text{order}=c(0,1,1),} \right.
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```r
code: text
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fit <- Arima(euretail, order=c(0,1,1), seasonal=c(0,1,1))

tsdisplay(residuals(fit))
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European quarterly retail trade

---

**Seasonal ARIMA models**
ACF and PACF of residuals show significant spikes at lag 2, and maybe lag 3.

ACIc of ARIMA(0,1,2)(0,1,1)4 model is 74.36.
ACIc of ARIMA(0,1,3)(0,1,1)4 model is 68.53.

```r
fit <- Arima(euretail, order=c(0,1,3),
             seasonal=c(0,1,1))
tsdisplay(residuals(fit))
Box.test(res, lag=16, fitdf=4,
         type="Ljung")
plot(forecast(fit3, h=12))
```
ACF and PACF of residuals show significant spikes at lag 2, and maybe lag 3.

AICₖ of ARIMA(0,1,2)(0,1,1)₄ model is 74.36.

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plot(forecast(fit3, h=12))
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European quarterly retail trade

Forecasting: Principles and Practice
Seasonal ARIMA models
Forecasts from ARIMA(0,1,3)(0,1,1)[4]
> auto.arima(euretail)
ARIMA(1,1,1)(0,1,1)[4]

Coefficients:

<table>
<thead>
<tr>
<th></th>
<th>ar1</th>
<th>ma1</th>
<th>sma1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.8828</td>
<td>-0.5208</td>
<td>-0.9704</td>
</tr>
<tr>
<td>s.e.</td>
<td>0.1424</td>
<td>0.1755</td>
<td>0.6792</td>
</tr>
</tbody>
</table>

sigma^2 estimated as 0.1411:  log likelihood=-30.19  
AIC=68.37  AICc=69.11  BIC=76.68
> auto.arima(euretail, stepwise=FALSE, 
  approximation=FALSE)
ARIMA(0,1,3)(0,1,1)[4]

Coefficients:

<table>
<thead>
<tr>
<th></th>
<th>ma1</th>
<th>ma2</th>
<th>ma3</th>
<th>sma1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.2625</td>
<td>0.3697</td>
<td>0.4194</td>
<td>-0.6615</td>
</tr>
<tr>
<td>s.e.</td>
<td>0.1239</td>
<td>0.1260</td>
<td>0.1296</td>
<td>0.1555</td>
</tr>
</tbody>
</table>

sigma^2 estimated as 0.1451: log likelihood=-28.7
AIC=67.4  AICc=68.53  BIC=77.78
Cortecosteroid drug sales

Year

H02 sales (million scripts)
1995 2000 2005
0.4 0.6 0.8 1.0 1.2

Year

Log H02 sales
1995 2000 2005
−1.0 −0.6 −0.2 0.2

Forecasting: Principles and Practice  Seasonal ARIMA models  22
Seasonally differenced H02 scripts

Year
1995 2000 2005

Lag
0 5 10 15 20 25 30 35

ACF

PACF

Forecasting: Principles and Practice
Seasonal ARIMA models
Choose $D = 1$ and $d = 0$.

- Spikes in PACF at lags 12 and 24 suggest seasonal AR(2) term.
- Spikes in PACF suggest possible non-seasonal AR(3) term.
- Initial candidate model: $\text{ARIMA}(3,0,0)(2,1,0)_{12}$. 
Choose $D = 1$ and $d = 0$.

Spikes in PACF at lags 12 and 24 suggest seasonal AR(2) term.

Spikes in PACF suggest possible non-seasonal AR(3) term.

Initial candidate model: $\text{ARIMA}(3,0,0)(2,1,0)_{12}$.
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Spikes in PACF at lags 12 and 24 suggest seasonal AR(2) term.

Spikes in PACF suggest possible non-seasonal AR(3) term.

Initial candidate model: \( \text{ARIMA}(3,0,0)(2,1,0)_{12} \).
Choose \( D = 1 \) and \( d = 0 \).

Spikes in PACF at lags 12 and 24 suggest seasonal AR(2) term.

Spikes in PACF suggest possible non-seasonal AR(3) term.

Initial candidate model: \( \text{ARIMA}(3,0,0)(2,1,0)_{12} \).
### Cortecosteroid drug sales

<table>
<thead>
<tr>
<th>Model</th>
<th>$\text{AIC}_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ARIMA(3,0,0)(2,1,0)$_{12}$</td>
<td>$-475.12$</td>
</tr>
<tr>
<td>ARIMA(3,0,1)(2,1,0)$_{12}$</td>
<td>$-476.31$</td>
</tr>
<tr>
<td>ARIMA(3,0,2)(2,1,0)$_{12}$</td>
<td>$-474.88$</td>
</tr>
<tr>
<td>ARIMA(3,0,1)(1,1,0)$_{12}$</td>
<td>$-463.40$</td>
</tr>
<tr>
<td>ARIMA(3,0,1)(0,1,1)$_{12}$</td>
<td>$-483.67$</td>
</tr>
<tr>
<td>ARIMA(3,0,1)(0,1,2)$_{12}$</td>
<td>$-485.48$</td>
</tr>
<tr>
<td>ARIMA(3,0,1)(1,1,1)$_{12}$</td>
<td>$-484.25$</td>
</tr>
</tbody>
</table>
Cortecosteroid drug sales

```
> fit <- Arima(h02, order=c(3,0,1),
  seasonal=c(0,1,2), lambda=0)

ARIMA(3,0,1)(0,1,2)[12]
Box Cox transformation: lambda= 0

Coefficients:

       ar1 ar2 ar3   ma1   sma1   sma2
-0.1603 0.5481 0.5678  0.3827 -0.5222 -0.1768

s.e. 0.1636 0.0878 0.0942  0.1895 0.0861 0.0872

sigma^2 estimated as 0.004145: log likelihood=250.04
AIC=-486.08   AICc=-485.48   BIC=-463.28
```
Corticosteroid drug sales

residuals(fit)

ACF

PACF

Lag

Lag
Cortecosteroid drug sales

tsdisplay(residuals(fit))
Box.test(residuals(fit), lag=36,
    fitdf=6, type="Ljung")
auto.arima(h02,lambda=0)
**Cortecosteroid drug sales**

**Training:** July 91 – June 06

**Test:** July 06 – June 08

<table>
<thead>
<tr>
<th>Model</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>ARIMA(3,0,0)(2,1,0)_{12}</td>
<td>0.0661</td>
</tr>
<tr>
<td>ARIMA(3,0,1)(2,1,0)_{12}</td>
<td>0.0646</td>
</tr>
<tr>
<td>ARIMA(3,0,2)(2,1,0)_{12}</td>
<td>0.0645</td>
</tr>
<tr>
<td>ARIMA(3,0,1)(1,1,0)_{12}</td>
<td>0.0679</td>
</tr>
<tr>
<td>ARIMA(3,0,1)(0,1,1)_{12}</td>
<td>0.0644</td>
</tr>
<tr>
<td>ARIMA(3,0,1)(0,1,2)_{12}</td>
<td>0.0622</td>
</tr>
<tr>
<td>ARIMA(3,0,1)(1,1,1)_{12}</td>
<td>0.0630</td>
</tr>
<tr>
<td>ARIMA(4,0,3)(0,1,1)_{12}</td>
<td>0.0648</td>
</tr>
<tr>
<td>ARIMA(3,0,3)(0,1,1)_{12}</td>
<td>0.0640</td>
</tr>
<tr>
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<td>0.0648</td>
</tr>
<tr>
<td>ARIMA(3,0,2)(0,1,1)_{12}</td>
<td>0.0644</td>
</tr>
<tr>
<td>ARIMA(2,1,3)(0,1,1)_{12}</td>
<td>0.0634</td>
</tr>
<tr>
<td>ARIMA(2,1,4)(0,1,1)_{12}</td>
<td>0.0632</td>
</tr>
<tr>
<td>ARIMA(2,1,5)(0,1,1)_{12}</td>
<td>0.0640</td>
</tr>
</tbody>
</table>
Cortecosteroid drug sales

**Training:** July 91 – June 06

**Test:** July 06 – June 08

<table>
<thead>
<tr>
<th>Model</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>ARIMA(3,0,0)(2,1,0)_{12}</td>
<td>0.0661</td>
</tr>
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<td>ARIMA(3,0,1)(2,1,0)_{12}</td>
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</tr>
<tr>
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</tr>
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<td><strong>0.0622</strong></td>
</tr>
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<td>0.0630</td>
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<tr>
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</tr>
<tr>
<td>ARIMA(2,1,5)(0,1,1)_{12}</td>
<td>0.0640</td>
</tr>
</tbody>
</table>
getrmse <- function(x,h,...)
{
    train.end <- time(x)[length(x)-h]
    test.start <- time(x)[length(x)-h+1]
    train <- window(x,end=train.end)
    test <- window(x,start=test.start)
    fit <- Arima(train,...)
    fc <- forecast(fit,h=h)
    return(accuracy(fc,test)[2,"RMSE"])
}
Cortecosteroid drug sales

getrmse(h02,h=24,order=c(3,0,0),seasonal=c(2,1,0),lambda=0)
getrmse(h02,h=24,order=c(3,0,1),seasonal=c(2,1,0),lambda=0)
getrmse(h02,h=24,order=c(3,0,2),seasonal=c(2,1,0),lambda=0)
getrmse(h02,h=24,order=c(3,0,1),seasonal=c(1,1,0),lambda=0)
getrmse(h02,h=24,order=c(3,0,1),seasonal=c(0,1,1),lambda=0)
getrmse(h02,h=24,order=c(3,0,1),seasonal=c(0,1,2),lambda=0)
getrmse(h02,h=24,order=c(3,0,1),seasonal=c(1,1,1),lambda=0)
getrmse(h02,h=24,order=c(4,0,3),seasonal=c(0,1,1),lambda=0)
getrmse(h02,h=24,order=c(3,0,3),seasonal=c(0,1,1),lambda=0)
getrmse(h02,h=24,order=c(4,0,2),seasonal=c(0,1,1),lambda=0)
getrmse(h02,h=24,order=c(3,0,2),seasonal=c(0,1,1),lambda=0)
getrmse(h02,h=24,order=c(2,1,3),seasonal=c(0,1,1),lambda=0)
getrmse(h02,h=24,order=c(2,1,4),seasonal=c(0,1,1),lambda=0)
getrmse(h02,h=24,order=c(2,1,5),seasonal=c(0,1,1),lambda=0)
Models with lowest $AIC_c$ values tend to give slightly better results than the other models.

$AIC_c$ comparisons must have the same orders of differencing. But RMSE test set comparisons can involve any models.

No model passes all the residual tests.

Use the best model available, even if it does not pass all tests.

In this case, the ARIMA$(3,0,1)(0,1,2)_{12}$ has the lowest RMSE value and the best $AIC_c$ value for models with fewer than 6 parameters.
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Cortecosteroid drug sales

- Models with lowest $AIC_c$ values tend to give slightly better results than the other models.
- $AIC_c$ comparisons must have the same orders of differencing. But RMSE test set comparisons can involve any models.
- **No model passes all the residual tests.**
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Forecasts from ARIMA(3,0,1)(0,1,2)[12]

Year
H02 sales (million scripts)
1995 2000 2005 2010
0.4 0.6 0.8 1.0 1.2 1.4 1.6
1. Backshift notation reviewed
2. Seasonal ARIMA models
3. ARIMA vs ETS
ARIMA vs ETS

- Myth that ARIMA models are more general than exponential smoothing.
- Linear exponential smoothing models all special cases of ARIMA models.
- Non-linear exponential smoothing models have no equivalent ARIMA counterparts.
- Many ARIMA models have no exponential smoothing counterparts.
- ETS models all non-stationary. Models with seasonality or non-damped trend (or both) have two unit roots; all other models have one unit root.
- Myth that ARIMA models are more general than exponential smoothing.
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Simple exponential smoothing

- Forecasts equivalent to \( \text{ARIMA}(0,1,1) \).
- Parameters: \( \theta_1 = \alpha - 1 \).

Holt’s method

- Forecasts equivalent to \( \text{ARIMA}(0,2,2) \).

Damped Holt’s method

Holt-Winters’ additive method

- No ARIMA equivalence

Holt-Winters’ multiplicative method
Equivalences

**Simple exponential smoothing**
- Forecasts equivalent to ARIMA(0,1,1).
- Parameters: $\theta_1 = \alpha - 1$.

**Holt’s method**
- Forecasts equivalent to ARIMA(0,2,2).
- Parameters: $\theta_1 = \alpha + \beta - 2$ and $\theta_2 = 1 - \alpha$.

**Damped Holt’s method**

**Holt-Winters’ additive method**
- Forecasts equivalent to ARIMA(0,1,m+1)(0,1,0)
- Parameter restrictions because ARIMA has $m+1$ parameters whereas HW uses only three parameters.

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- Forecasts equivalent to $\text{ARIMA}(1,1,2)$.

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Equivalences

Simple exponential smoothing

- Forecasts equivalent to \textbf{ARIMA(0,1,1)}.
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- Forecasts equivalent to \textbf{ARIMA(0,2,2)}.
- Parameters: $\theta_1 = \alpha + \beta - 2$ and $\theta_2 = 1 - \alpha$.

Damped Holt’s method

- Forecasts equivalent to \textbf{ARIMA(1,1,2)}.
- Parameters: $\phi_1 = \phi$, $\theta_1 = \alpha + \phi \beta - 2$, $\theta_2 = (1 - \alpha) \phi$.

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Holt-Winters’ multiplicative method
Equivalences

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- Forecasts equivalent to $\text{ARIMA}(0,1,1)$.
- Parameters: $\theta_1 = \alpha - 1$.

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- Forecasts equivalent to \textbf{ARIMA}(0,1,m+1)(0,1,0)_m.

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- Parameter restrictions because \( \text{ARIMA} \) has \( m + 1 \) parameters whereas \( \text{HW} \) uses only three parameters.

Holt-Winters’ multiplicative method

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