MONASH University

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Forecasting: Principles and Practice

8. Seasonal ARIMA models

OTexts.com/fpp/8/9

Outline

Backshift notation

A very useful notational device is the backward shift operator, B, which is used as follows:

$$By_t = y_{t-1} \, .$$

In other words, B, operating on y_t, has the effect of **shifting the data back one period**. Two applications of B to y_t **shifts the data back two periods**:

$$B(By_t) = B^2 y_t = y_{t-2} \, .$$

For monthly data, if we wish to shift attention to "the same month last year," then B^{12} is used, and the notation is $B^{12} y_t = y_{t-12}$.

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- First difference: $1 - B$.
- Double difference: $(1 - B)^2$.
- dth-order difference: $(1 - B)^d y_t$.
- Seasonal difference: $1 - B^m$.
- Seasonal difference followed by a first difference: $(1 - B)(1 - B^m)$.
- Multiply terms together together to see the combined effect:

$$\begin{aligned}(1 - B)(1 - B^m)y_t &= (1 - B - B^m + B^{m+1})y_t \\ &= y_t - y_{t-1} - y_{t-m} + y_{t-m-1}.\end{aligned}$$

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Backshift notation for ARIMA

- ARMA model:

$$\begin{aligned} y_t &= c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + e_t + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q} \\ &= c + \phi_1 B y_t + \cdots + \phi_p B^p y_t + e_t + \theta_1 B e_t + \cdots + \theta_q B^q e_t \end{aligned}$$

$$\phi(B) y_t = c + \theta(B) e_t$$

$$\text{where } \phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$$

$$\text{and } \theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q.$$

- ARIMA(1,1,1) model:

$$(1 - \phi_1 B) \; (1 - B) y_t \; = \; c + (1 + \theta_1 B) e_t$$

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- ARIMA(1,1,1) model:

$$(1 - \phi_1 B) \quad (1 - B) y_t \quad = \quad c + (1 + \underset{\substack{\uparrow \\ \text{MA(1)}}}{\theta_1 B}) e_t$$

Outline

1 Backshift notation reviewed

2 Seasonal ARIMA models

3 ARIMA vs ETS

Seasonal ARIMA models

$$\text{ARIMA}\ \ (p, d, q)\ \ (P, D, Q)_m$$

where $m =$ number of periods per season.

Seasonal ARIMA models

$$\text{ARIMA} \quad \underbrace{(p,d,q)}_{\uparrow} \quad (P,D,Q)_m$$

$$\left(\begin{array}{l}\text{Non-}\\\text{seasonal}\\\text{part of the}\\\text{model}\end{array}\right)$$

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Seasonal ARIMA models

E.g., ARIMA$(1,1,1)(1,1,1)_4$ model (without constant)

$$(1-\phi_1 B)(1-\Phi_1 B^4)(1-B)(1-B^4)y_t \;=\; (1+\theta_1 B)(1+\Theta_1 B^4)e_t.$$

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$$(1-\phi_1 B)\underset{\uparrow \atop \begin{pmatrix}\text{Seasonal}\\ \text{AR(1)}\end{pmatrix}}{(1-\Phi_1 B^4)}(1-B)(1-B^4)y_t = (1+\theta_1 B)(1+\Theta_1 B^4)e_t.$$

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All the factors can be multiplied out and the general model written as follows:

$$\begin{aligned}
y_t = {} & (1+\phi_1)y_{t-1} - \phi_1 y_{t-2} + (1+\Phi_1)y_{t-4} \\
& - (1+\phi_1+\Phi_1+\phi_1\Phi_1)y_{t-5} + (\phi_1+\phi_1\Phi_1)y_{t-6} \\
& - \Phi_1 y_{t-8} + (\Phi_1+\phi_1\Phi_1)y_{t-9} - \phi_1\Phi_1 y_{t-10} \\
& + e_t + \theta_1 e_{t-1} + \Theta_1 e_{t-4} + \theta_1\Theta_1 e_{t-5}.
\end{aligned}$$

Common ARIMA models

In the US Census Bureau uses the following models most often:

ARIMA$(0,1,1)(0,1,1)_m$	with log transformation
ARIMA$(0,1,2)(0,1,1)_m$	with log transformation
ARIMA$(2,1,0)(0,1,1)_m$	with log transformation
ARIMA$(0,2,2)(0,1,1)_m$	with log transformation
ARIMA$(2,1,2)(0,1,1)_m$	with no transformation

Seasonal ARIMA models

The seasonal part of an AR or MA model will be seen in the seasonal lags of the PACF and ACF.

ARIMA(0,0,0)(0,0,1)$_{12}$ will show:

- a spike at lag 12 in the ACF but no other significant spikes.
- The PACF will show exponential decay in the seasonal lags; that is, at lags 12, 24, 36, ...

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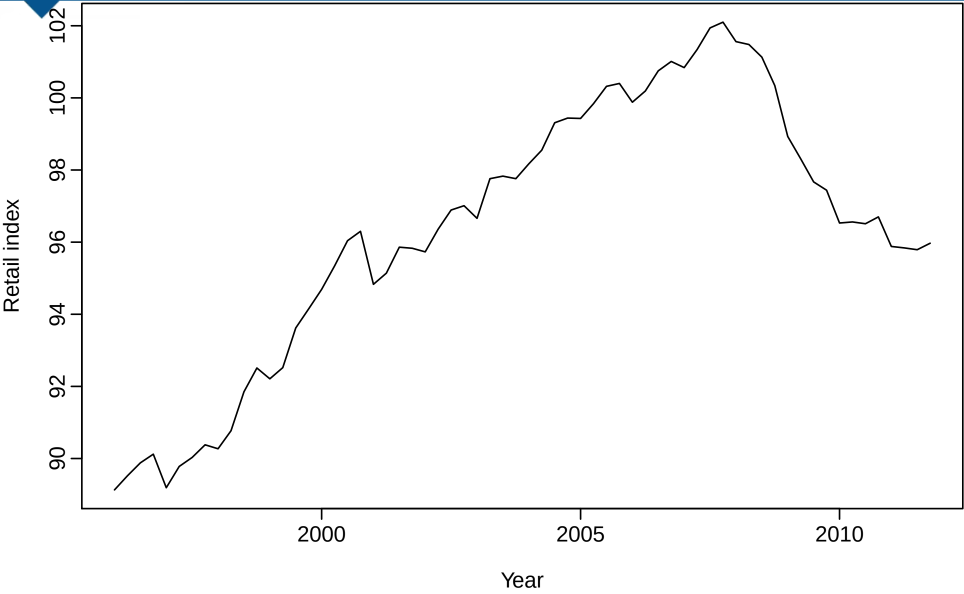
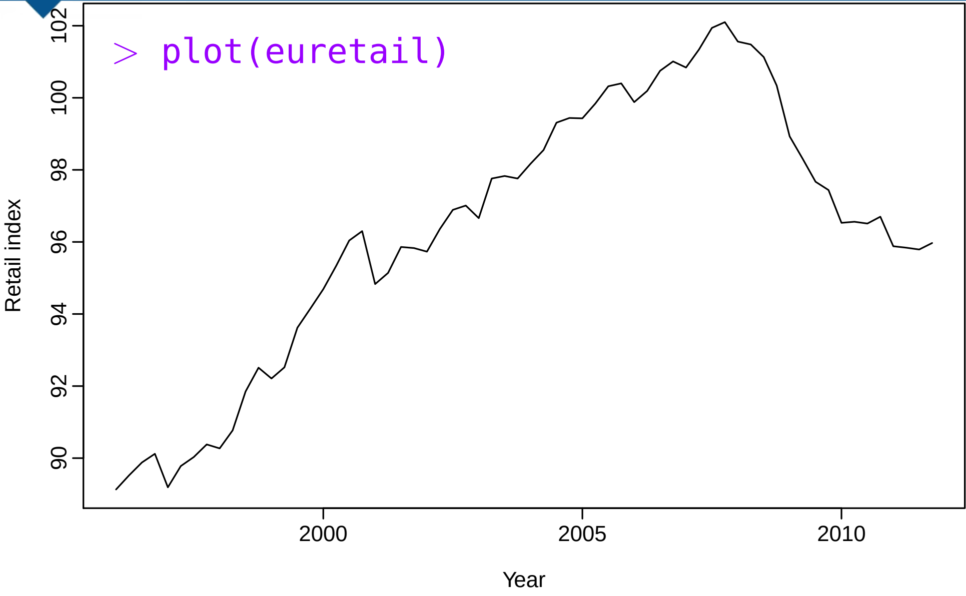
European quarterly retail trade



European quarterly retail trade

```
> plot(euretail)
```



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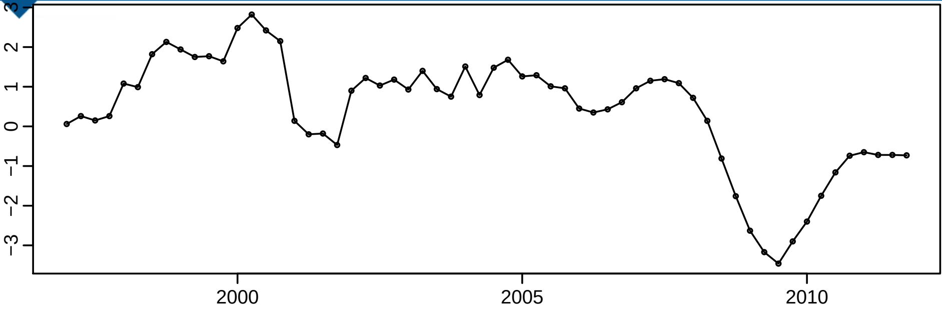

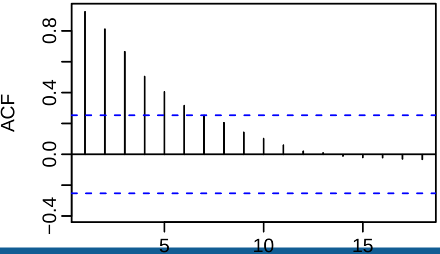

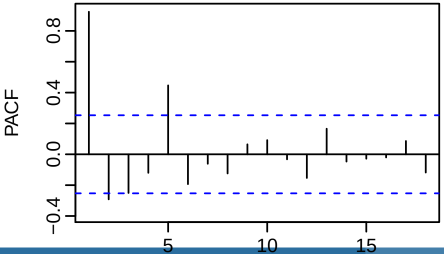

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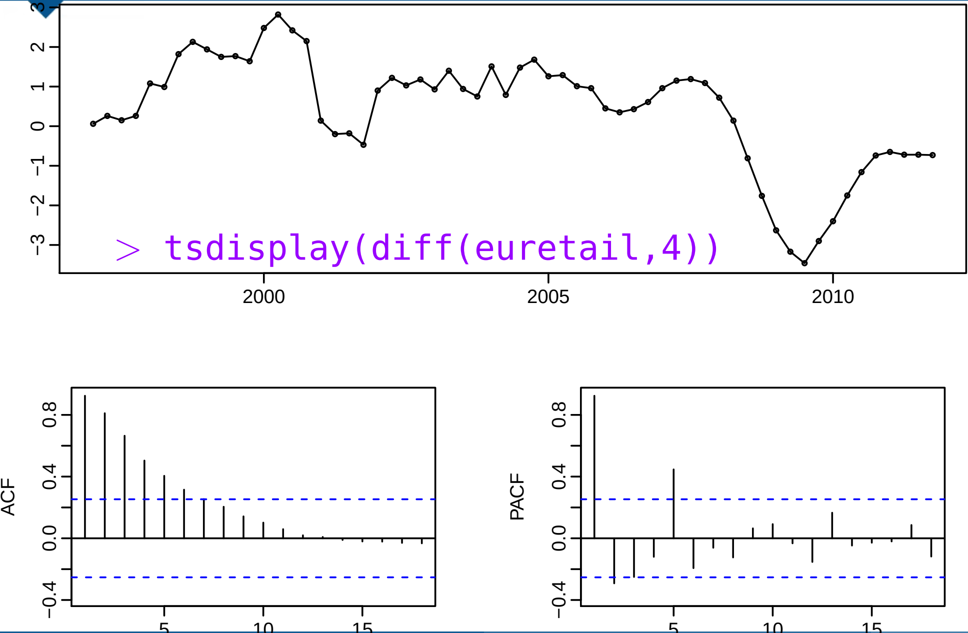


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> tsdisplay(diff(euretail,4))
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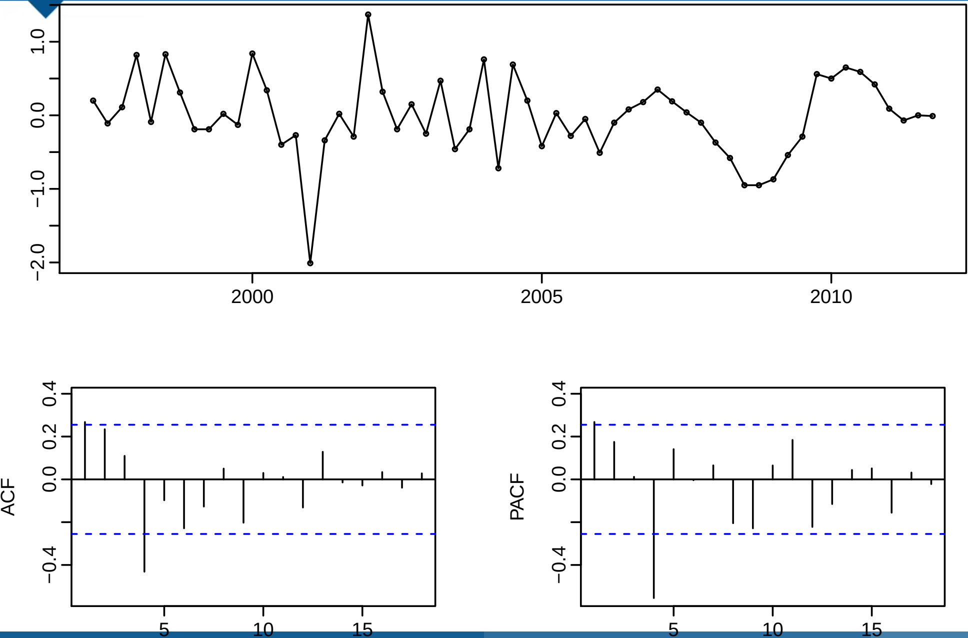

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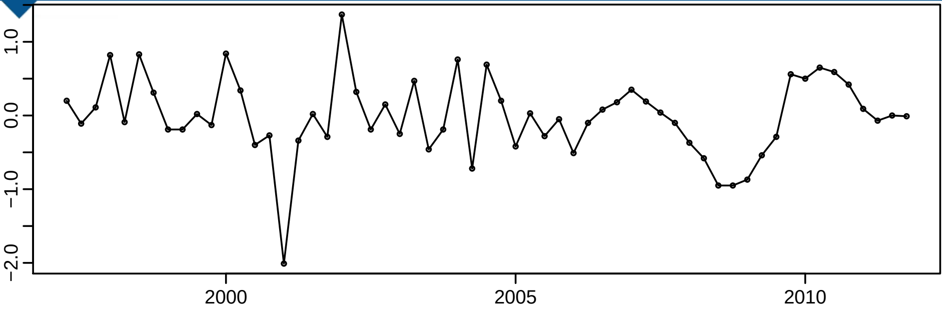


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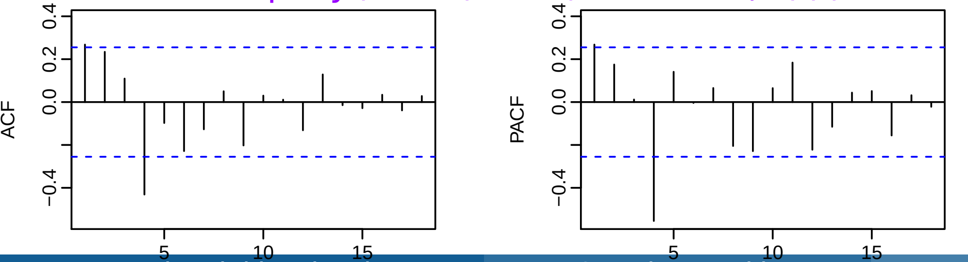

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- $d = 1$ and $D = 1$ seems necessary.
- Significant spike at lag 1 in ACF suggests non-seasonal MA(1) component.
- Significant spike at lag 4 in ACF suggests seasonal MA(1) component.
- Initial candidate model: ARIMA(0,1,1)(0,1,1)$_4$.
- We could also have started with ARIMA(1,1,0)(1,1,0)$_4$.

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fit <- Arima(euretail, order=c(0,1,1),
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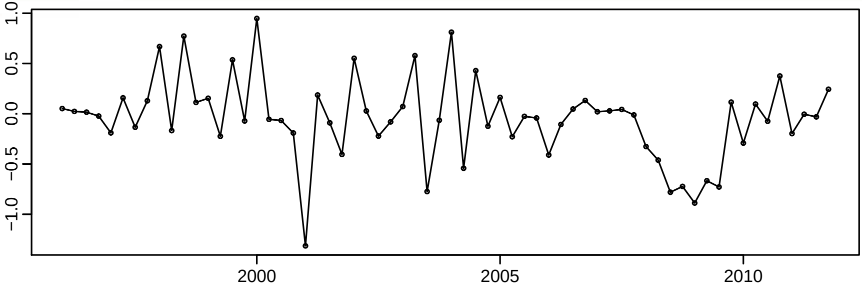

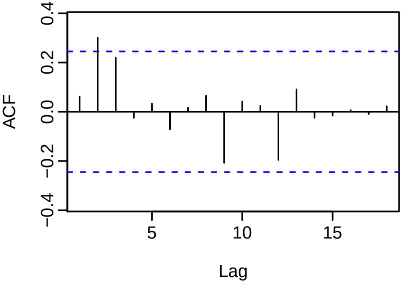

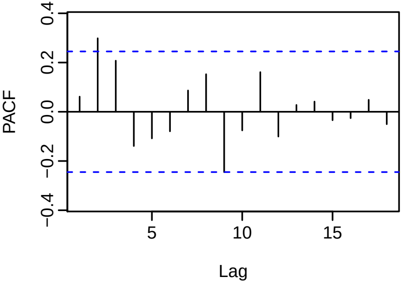

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- ACF and PACF of residuals show significant spikes at lag 2, and maybe lag 3.
- AIC_c of ARIMA(0,1,2)$(0,1,1)_4$ model is 74.36.
- AIC_c of ARIMA(0,1,3)$(0,1,1)_4$ model is 68.53.

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fit <- Arima(euretail, order=c(0,1,3),
  seasonal=c(0,1,1))
tsdisplay(residuals(fit))
Box.test(res, lag=16, fitdf=4,
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plot(forecast(fit3, h=12))
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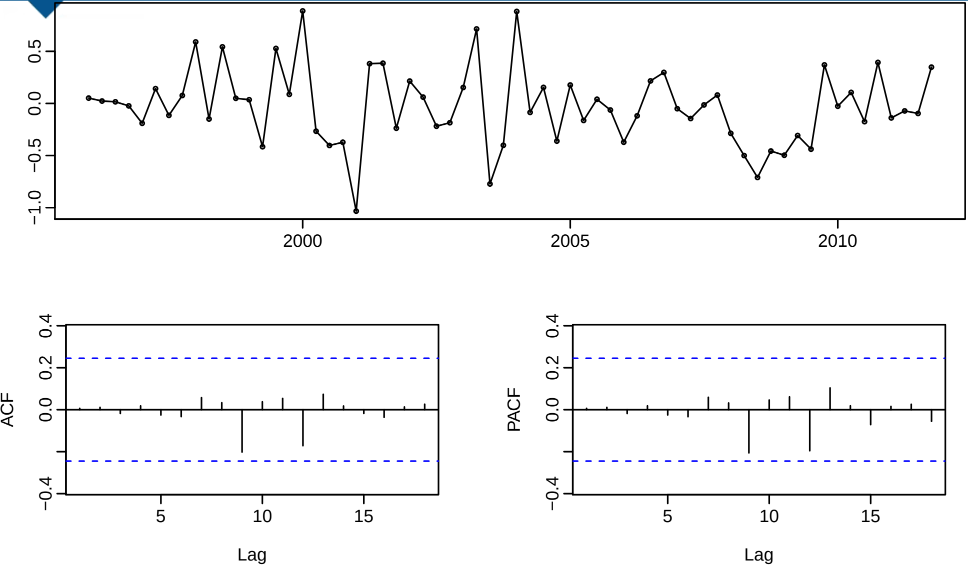

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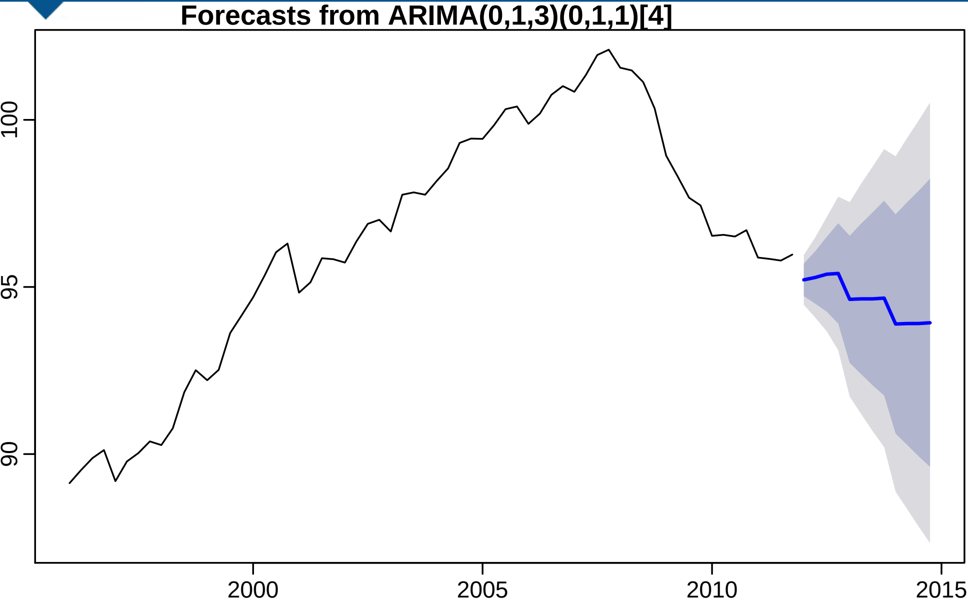


European quarterly retail trade

```
> auto.arima(euretail)
ARIMA(1,1,1)(0,1,1)[4]

Coefficients:
         ar1      ma1     sma1
      0.8828  -0.5208  -0.9704
s.e.  0.1424   0.1755   0.6792

sigma^2 estimated as 0.1411:  log likelihood=-30.19
AIC=68.37   AICc=69.11   BIC=76.68
```

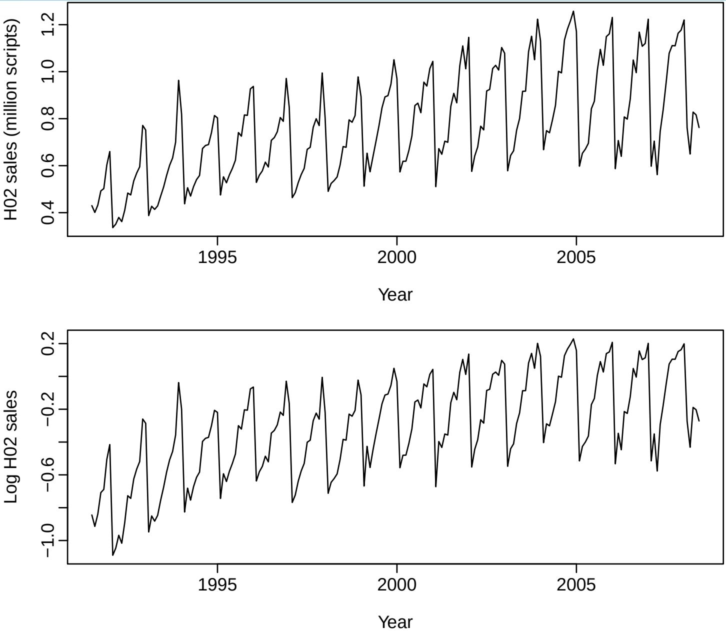
European quarterly retail trade

```
> auto.arima(euretail, stepwise=FALSE,
    approximation=FALSE)
ARIMA(0,1,3)(0,1,1)[4]

Coefficients:
         ma1     ma2     ma3     sma1
      0.2625  0.3697  0.4194  -0.6615
s.e.  0.1239  0.1260  0.1296   0.1555

sigma^2 estimated as 0.1451:  log likelihood=-28.7
AIC=67.4   AICc=68.53   BIC=77.78
```

Cortecosteroid drug sales



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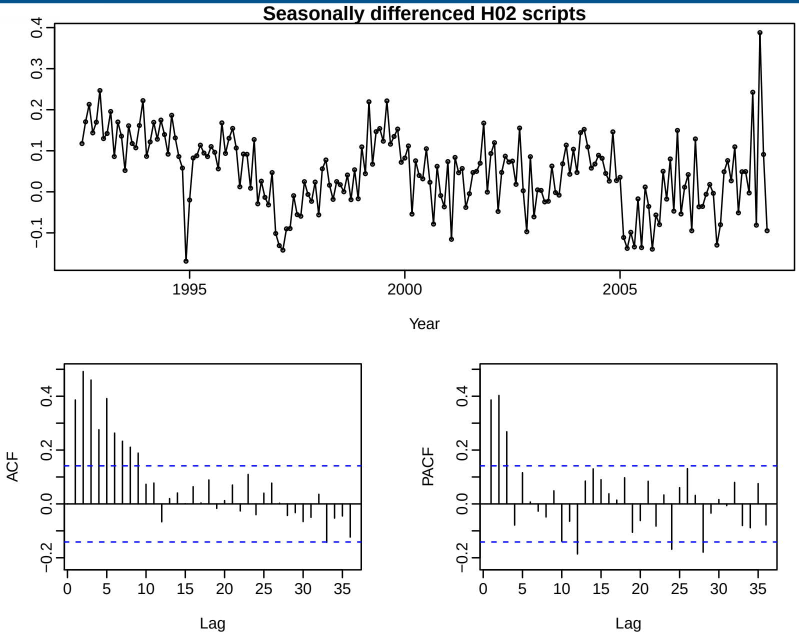

Cortecosteroid drug sales

- Choose $D = 1$ and $d = 0$.
- Spikes in PACF at lags 12 and 24 suggest seasonal AR(2) term.
- Spikes in PACF sugges possible non-seasonal AR(3) term.
- Initial candidate model: ARIMA(3,0,0)(2,1,0)$_{12}$.

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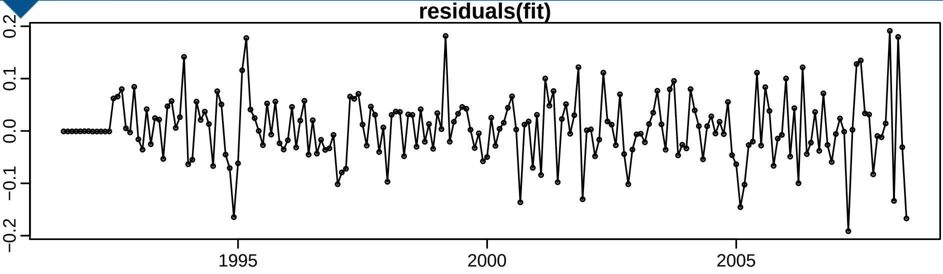
Cortecosteroid drug sales

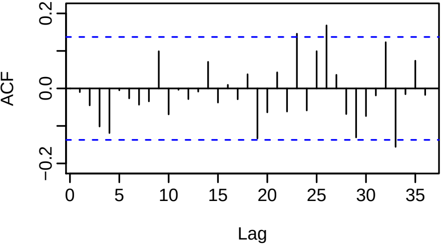
Model	AIC_c
$\text{ARIMA}(3,0,0)(2,1,0)_{12}$	-475.12
$\text{ARIMA}(3,0,1)(2,1,0)_{12}$	-476.31
$\text{ARIMA}(3,0,2)(2,1,0)_{12}$	-474.88
$\text{ARIMA}(3,0,1)(1,1,0)_{12}$	-463.40
$\text{ARIMA}(3,0,1)(0,1,1)_{12}$	-483.67
$\text{ARIMA}(3,0,1)(0,1,2)_{12}$	-485.48
$\text{ARIMA}(3,0,1)(1,1,1)_{12}$	-484.25

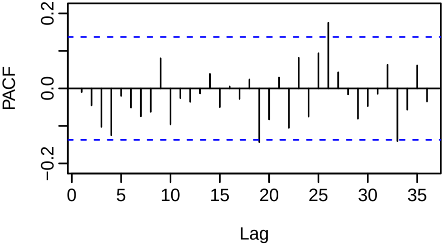
Cortecosteroid drug sales

```
> fit <- Arima(h02, order=c(3,0,1),
  seasonal=c(0,1,2), lambda=0)

ARIMA(3,0,1)(0,1,2)[12]
Box Cox transformation: lambda= 0

Coefficients:
          ar1     ar2     ar3     ma1     sma1     sma2
      -0.1603  0.5481  0.5678  0.3827  -0.5222  -0.1768
s.e.   0.1636  0.0878  0.0942  0.1895   0.0861   0.0872

sigma^2 estimated as 0.004145:  log likelihood=250.04
AIC=-486.08   AICc=-485.48   BIC=-463.28
```

Cortecosteroid drug sales







Cortecosteroid drug sales

```
tsdisplay(residuals(fit))
Box.test(residuals(fit), lag=36,
  fitdf=6, type="Ljung")
auto.arima(h02,lambda=0)
```

Cortecosteroid drug sales

Training: July 91 – June 06

Test: July 06 – June 08

Model	RMSE
ARIMA$(3,0,0)(2,1,0)_{12}$	0.0661
ARIMA$(3,0,1)(2,1,0)_{12}$	0.0646
ARIMA$(3,0,2)(2,1,0)_{12}$	0.0645
ARIMA$(3,0,1)(1,1,0)_{12}$	0.0679
ARIMA$(3,0,1)(0,1,1)_{12}$	0.0644
ARIMA$(3,0,1)(0,1,2)_{12}$	0.0622
ARIMA$(3,0,1)(1,1,1)_{12}$	0.0630
ARIMA$(4,0,3)(0,1,1)_{12}$	0.0648
ARIMA$(3,0,3)(0,1,1)_{12}$	0.0640
ARIMA$(4,0,2)(0,1,1)_{12}$	0.0648
ARIMA$(3,0,2)(0,1,1)_{12}$	0.0644
ARIMA$(2,1,3)(0,1,1)_{12}$	0.0634
ARIMA$(2,1,4)(0,1,1)_{12}$	0.0632
ARIMA$(2,1,5)(0,1,1)_{12}$	0.0640

Cortecosteroid drug sales

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ARIMA(2,1,5)(0,1,1)$_{12}$	0.0640

Cortecosteroid drug sales

```
getrmse <- function(x,h,...)
{
  train.end <- time(x)[length(x)-h]
  test.start <- time(x)[length(x)-h+1]
  train <- window(x,end=train.end)
  test <- window(x,start=test.start)
  fit <- Arima(train,...)
  fc <- forecast(fit,h=h)
  return(accuracy(fc,test)[2,"RMSE"])
}
```

Cortecosteroid drug sales

```
getrmse(h02,h=24,order=c(3,0,0),seasonal=c(2,1,0),lambda=0)
getrmse(h02,h=24,order=c(3,0,1),seasonal=c(2,1,0),lambda=0)
getrmse(h02,h=24,order=c(3,0,2),seasonal=c(2,1,0),lambda=0)
getrmse(h02,h=24,order=c(3,0,1),seasonal=c(1,1,0),lambda=0)
getrmse(h02,h=24,order=c(3,0,1),seasonal=c(0,1,1),lambda=0)
getrmse(h02,h=24,order=c(3,0,1),seasonal=c(0,1,2),lambda=0)
getrmse(h02,h=24,order=c(3,0,1),seasonal=c(1,1,1),lambda=0)
getrmse(h02,h=24,order=c(4,0,3),seasonal=c(0,1,1),lambda=0)
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getrmse(h02,h=24,order=c(2,1,3),seasonal=c(0,1,1),lambda=0)
getrmse(h02,h=24,order=c(2,1,4),seasonal=c(0,1,1),lambda=0)
getrmse(h02,h=24,order=c(2,1,5),seasonal=c(0,1,1),lambda=0)
```

Cortecosteroid drug sales

- Models with lowest AIC_c values tend to give slightly better results than the other models.
- AIC_c comparisons must have the same orders of differencing. But RMSE test set comparisons can involve any models.
- No model passes all the residual tests.
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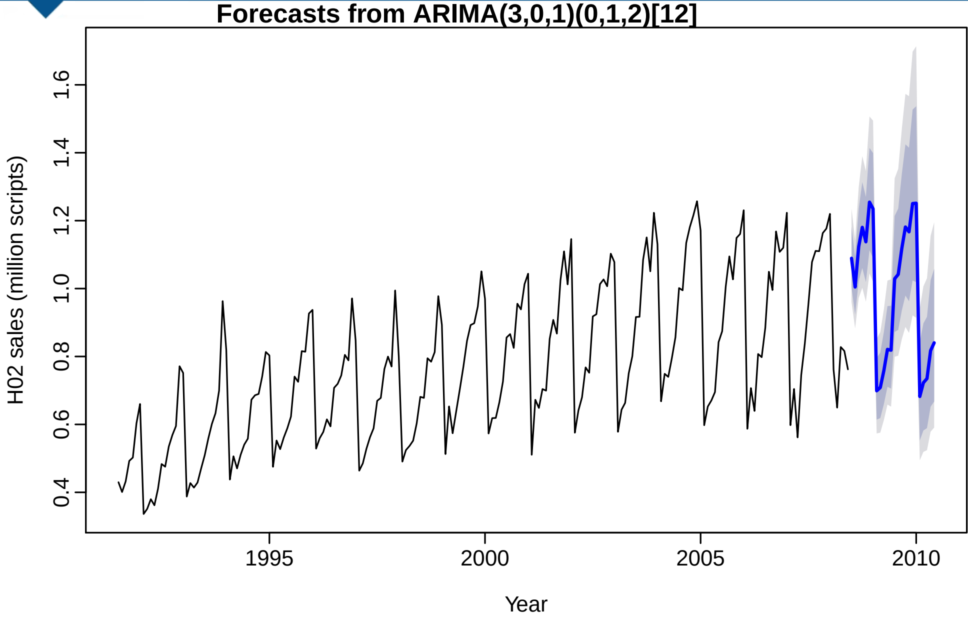

Outline

1 Backshift notation reviewed

2 Seasonal ARIMA models

3 ARIMA vs ETS

ARIMA vs ETS

- Myth that ARIMA models are more general than exponential smoothing.
- Linear exponential smoothing models all special cases of ARIMA models.
- Non-linear exponential smoothing models have no equivalent ARIMA counterparts.
- Many ARIMA models have no exponential smoothing counterparts.
- ETS models all non-stationary. Models with seasonality or non-damped trend (or both) have two unit roots; all other models have one unit root.

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Equivalences

Simple exponential smoothing

- Forecasts equivalent to **ARIMA(0,1,1)**.
- Parameters: $\theta_1 = \alpha - 1$.

Holt's method

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Damped Holt's method

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- Parameters: $\theta_1 = \alpha + \beta - 2$ and $\theta_2 = 1 - \alpha$.

Damped Holt's method

- Forecasts equivalent to **ARIMA(1,1,2)**.
- Parameters: $\phi_1 = \phi$, $\theta_1 = \alpha + \phi\beta - 2$, $\theta_2 = (1 - \alpha)\phi$.

Holt-Winters' additive method

- Forecasts equivalent to **ARIMA(0,1,m+1)(0,1,0)$_\mathbf{m}$**.
- Parameter restrictions because ARIMA has $m + 1$ parameters whereas HW uses only three parameters.

Holt-Winters' multiplicative method

- No ARIMA equivalence

Equivalences

Simple exponential smoothing

- Forecasts equivalent to **ARIMA(0,1,1)**.
- Parameters: $\theta_1 = \alpha - 1$.

Holt's method

- Forecasts equivalent to **ARIMA(0,2,2)**.
- Parameters: $\theta_1 = \alpha + \beta - 2$ and $\theta_2 = 1 - \alpha$.

Damped Holt's method

- Forecasts equivalent to **ARIMA(1,1,2)**.
- Parameters: $\phi_1 = \phi$, $\theta_1 = \alpha + \phi\beta - 2$, $\theta_2 = (1 - \alpha)\phi$.

Holt-Winters' additive method

- Forecasts equivalent to **ARIMA(0,1,m+1)(0,1,0)$_{\mathbf{m}}$**.
- Parameter restrictions because ARIMA has $m + 1$ parameters whereas HW uses only three parameters.

Holt-Winters' multiplicative method

- No ARIMA equivalence