MONASH University

Rob J Hyndman

Forecasting:

Principles and Practice

10. Dynamic regression

OTexts.com/fpp/9/1/

Outline

Regression with ARIMA errors

Regression models

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + e_t,$$

- y_t modeled as function of k explanatory variables $x_{1,t}, \dots, x_{k,t}$.
- Previously, we assumed that e_t was WN.
- Now we want to allow e_t to be autocorrelated.

Example: ARIMA(1,1,1) errors

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + n_t,$$

$$(1 - \phi_1 B)(1 - B) n_t = (1 + \theta_1 B) e_t,$$

where e_t is white noise .

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Estimation

If we minimize $\sum n_t^2$ (by using ordinary regression):

1. Estimated coefficients $\hat{\beta}_0, \dots, \hat{\beta}_k$ are no longer optimal as some information ignored;
2. Statistical tests associated with the model (e.g., t-tests on the coefficients) are incorrect.
3. p-values for coefficients usually too small ("spurious regression").
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Stationarity

Regression with ARMA errors

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + n_t,$$

where n_t is an ARMA process.

- All variables in the model must be stationary.
- If we estimate the model while any of these are non-stationary, the estimated coefficients can be incorrect.
- Difference variables until all stationary.
- If necessary, apply same differencing to all variables.

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Equivalent to model with ARIMA(1,0,1) errors

$$y'_t = \beta_1 x'_{1,t} + \dots + \beta_k x'_{k,t} + n'_t,$$
$$(1 - \phi_1 B)n'_t = (1 + \theta_1 B)e_t,$$

where $y'_t = y_t - y_{t-1}$, $x'_{t,i} = x_{t,i} - x_{t-1,i}$ and $n'_t = n_t - n_{t-1}$.

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Regression with ARIMA errors

Any regression with an ARIMA error can be rewritten as a regression with an ARMA error by differencing all variables with the same differencing operator as in the ARIMA model.

Original data

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + n_t$$

$$\text{where} \quad \phi(B)(1-B)^d N_t = \theta(B)e_t$$

After differencing all variables

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Model selection

- To determine ARIMA error structure, first need to calculate n_t.
- We can't get n_t without knowing $\beta_0, \dots, \beta_k$.
- To estimate these, we need to specify ARIMA error structure.

Solution: Begin with a proxy model for the ARIMA errors.

Estimate model, determine better error structure, and re-estimate.

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- Assume ARIMA(2,0,0)(1,0,0)$_m$ model for seasonal data.

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- AIC can be calculated for final model.
- Repeat procedure for all subsets of predictors to be considered, and select model with lowest AIC value.

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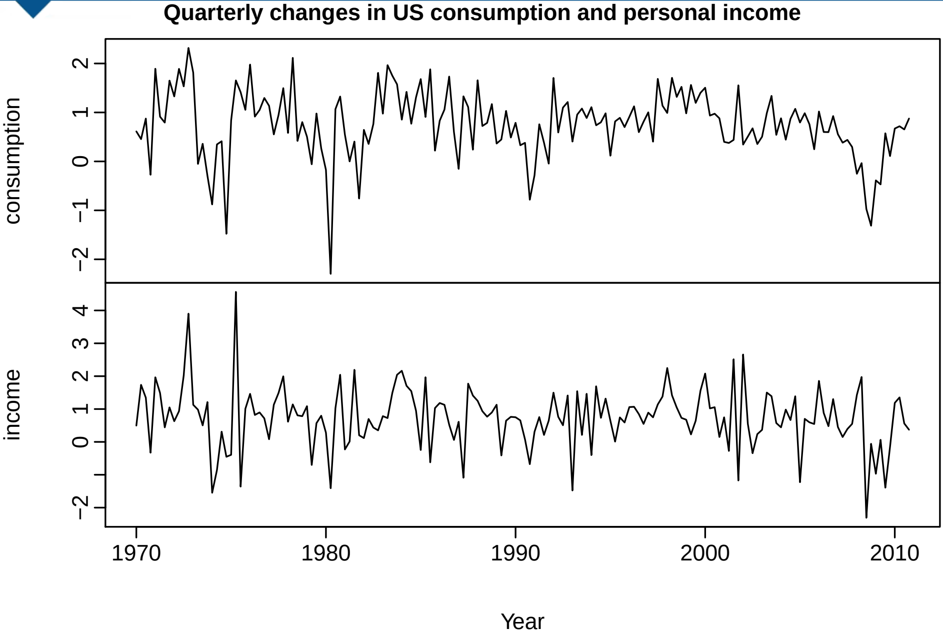
US personal consumption & income



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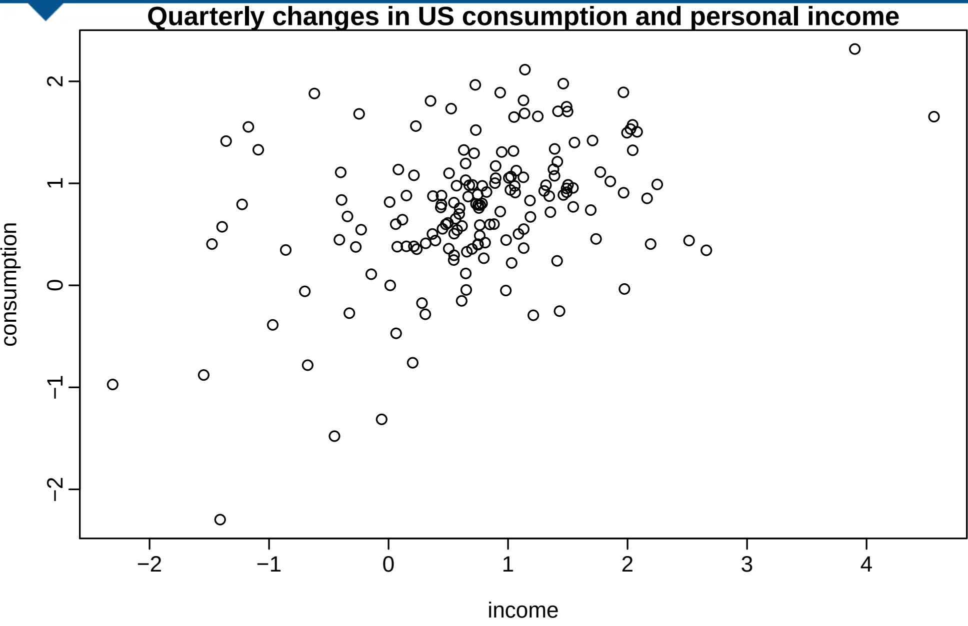

US Personal Consumption and Income

- No need for transformations or further differencing.
- Increase in income does not necessarily translate into instant increase in consumption (e.g., after the loss of a job, it may take a few months for expenses to be reduced to allow for the new circumstances). We will ignore this for now.
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fit <- Arima(usconsumption[,1],
             xreg=usconsumption[,2],
             order=c(2,0,0))
tsdisplay(arima.errors(fit),
             main="ARIMA errors")
```

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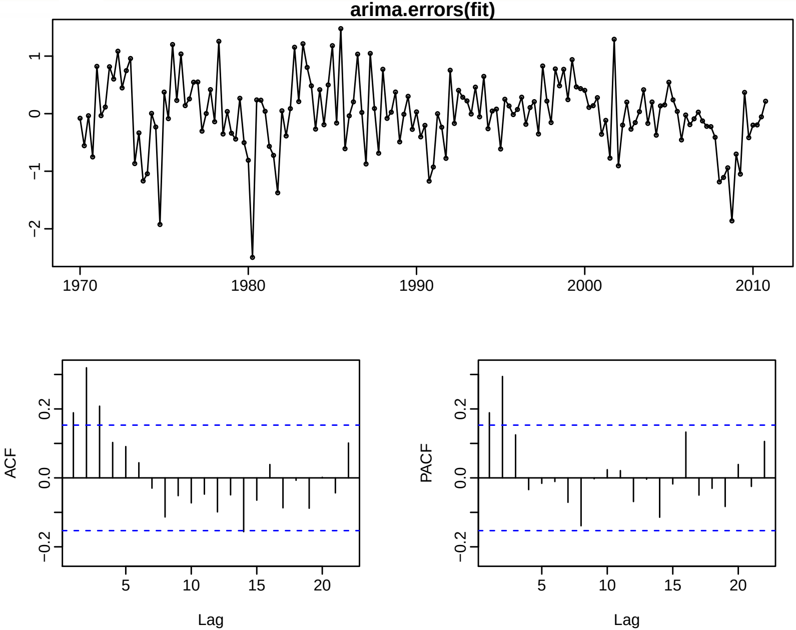

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- Candidate ARIMA models include MA(3) and AR(2).
- ARIMA(1,0,2) has lowest AIC_c value.
- Refit model with ARIMA(1,0,2) errors.

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Coefficients:
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s.e. 0.1468  0.1576 0.0790    0.0951    0.0513

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  xreg=usconsumption[,2], order=c(1,0,2)))

Coefficients:
        ar1      ma1     ma2  intercept  usconsumption[,2]
     0.6516  -0.5440  0.2187     0.5750             0.2420
s.e. 0.1468   0.1576  0.0790     0.0951             0.0513

sigma^2 estimated as 0.3396:  log likelihood=-144.27
AIC=300.54   AICc=301.08   BIC=319.14
```

US Personal Consumption and Income

The whole process can be automated:

```
> auto.arima(usconsumption[,1], xreg=usconsumption[,2])
Series: usconsumption[, 1]
ARIMA(1,0,2) with non-zero mean

Coefficients:
        ar1     ma1    ma2 intercept usconsumption[,2]
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```

US Personal Consumption and Income

```
> Box.test(residuals(fit2), fitdf=5,
       lag=10, type="Ljung")

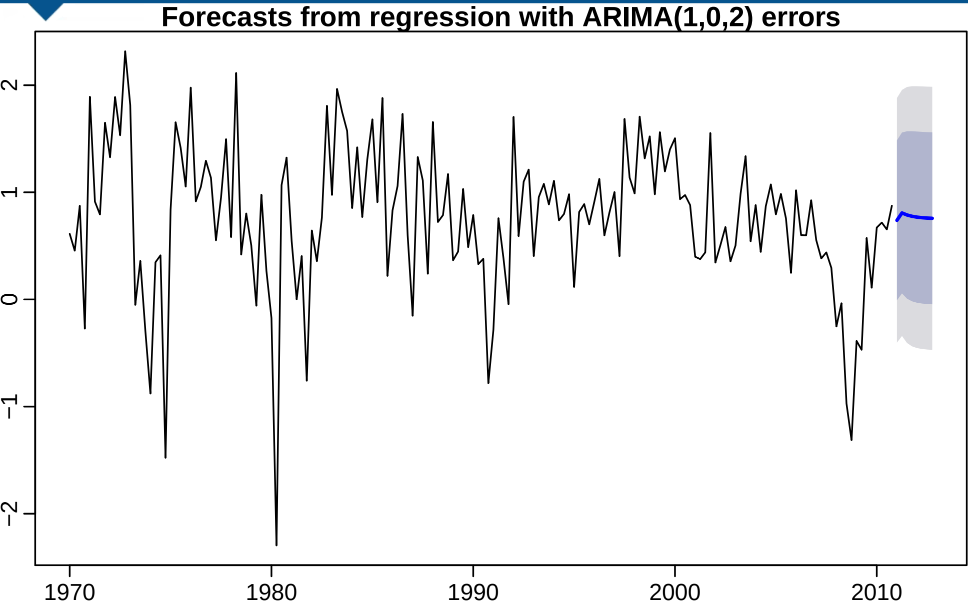
  Box-Ljung test
data:  residuals(fit2)
X-squared = 4.5948, df = 5, p-value = 0.4673
```

US Personal Consumption and Income

```
fcast <- forecast(fit2,
  xreg=rep(mean(usconsumption[,2]),8), h=8)

plot(fcast,
  main="Forecasts from regression with
    ARIMA(1,0,2) errors")
```

US Personal Consumption and Income



Forecasting

- To forecast a regression model with ARIMA errors, we need to forecast the regression part of the model and the ARIMA part of the model and combine the results.
- Forecasts of macroeconomic variables may be obtained from the ABS, for example.
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Outline

Stochastic & deterministic trends

Deterministic trend

$$y_t = \beta_0 + \beta_1 t + n_t$$

where n_t is ARMA process.

Stochastic trend

$$y_t = \beta_0 + \beta_1 t + n_t$$

where n_t is ARIMA process with $d \geq 1$.
Difference both sides until n_t is stationary:

$$y'_t = \beta_1 + n'_t$$

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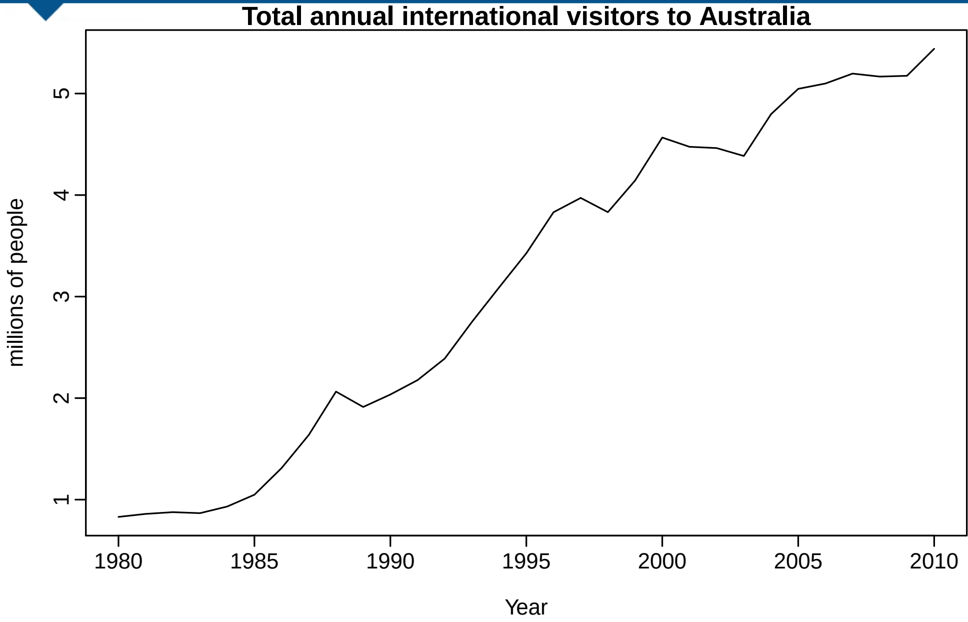
International visitors



International visitors

Deterministic trend

```
> auto.arima(austa,d=0,xreg=1:length(austa))
ARIMA(2,0,0) with non-zero mean

Coefficients:
         ar1      ar2  intercept  1:length(austa)
      1.0371  -0.3379     0.4173           0.1715
s.e.  0.1675   0.1797     0.1866           0.0102

sigma^2 estimated as 0.02486:  log likelihood=12.7
AIC=-15.4   AICc=-13   BIC=-8.23
```

$$y_t = 0.4173 + 0.1715t + n_t$$
$$n_t = 1.0371n_{t-1} - 0.3379n_{t-2} + e_t$$
$$e_t \sim \text{NID}(0, 0.02486).$$

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International visitors

Stochastic trend

```
> auto.arima(austa,d=1)
ARIMA(0,1,0) with drift

Coefficients:
       drift
      0.1538
s.e.  0.0323

sigma^2 estimated as 0.03132:  log likelihood=9.38
AIC=-14.76   AICc=-14.32   BIC=-11.96
```

$$y_t - y_{t-1} = 0.1538 + e_t$$

$$y_t = y_0 + 0.1538t + n_t$$

$$n_t = n_{t-1} + e_t$$

$$e_t \sim \text{NID}(0, 0.03132).$$

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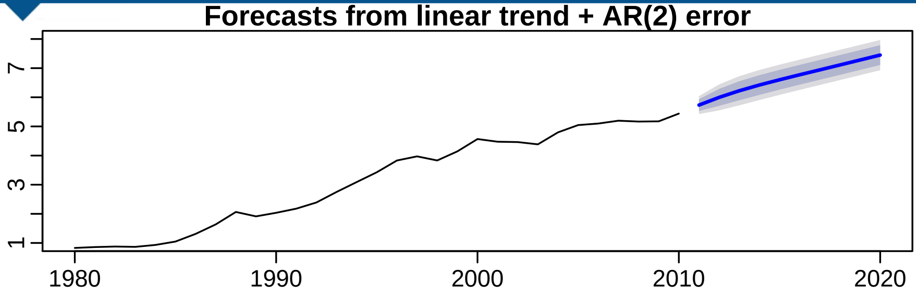

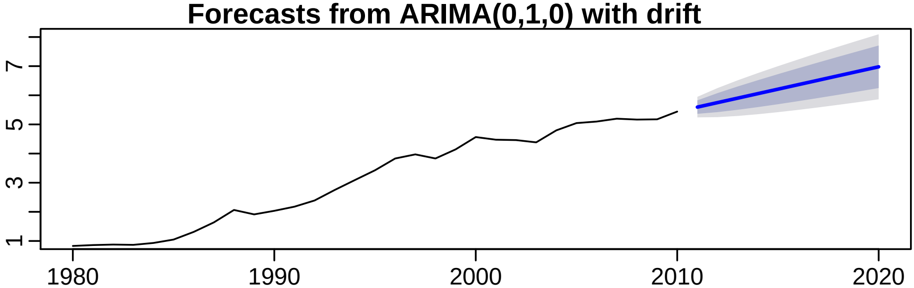

Forecasting with trend

- Point forecasts are almost identical, but prediction intervals differ.
- Stochastic trends have much wider prediction intervals because the errors are non-stationary.
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Outline

Fourier terms for seasonality

Periodic seasonality can be handled using pairs of Fourier terms:

$$s_k(t) = \sin\left(\frac{2\pi kt}{m}\right) \qquad c_k(t) = \cos\left(\frac{2\pi kt}{m}\right)$$

$$y_t = \sum_{k=1}^{K} [\alpha_k s_k(t) + \beta_k c_k(t)] + n_t$$

- n_t is non-seasonal ARIMA process.
- Every periodic function can be approximated by sums of sin and cos terms for large enough K.
- Choose K by minimizing AICc.

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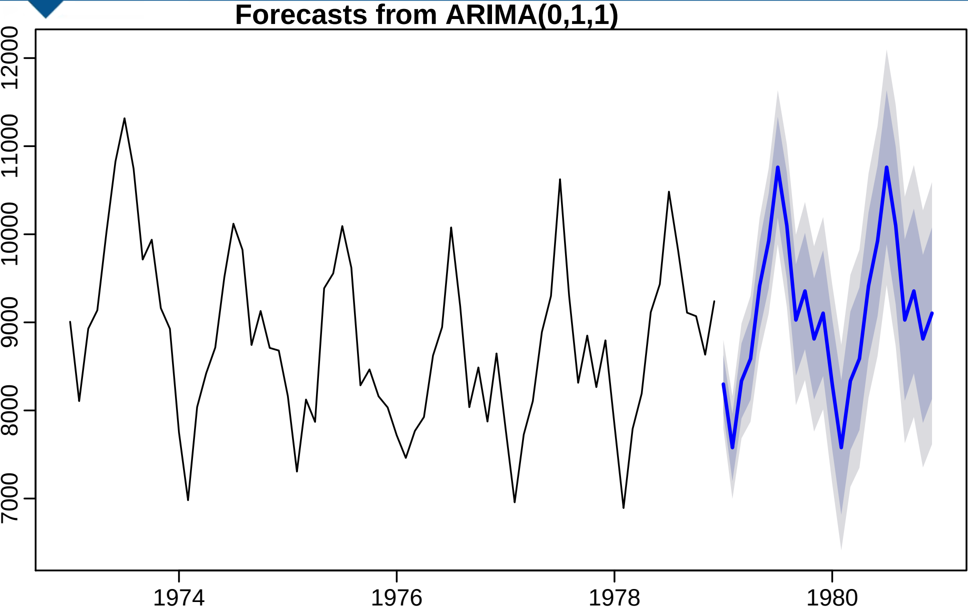
US Accidental Deaths

```
fit <- auto.arima(USAccDeaths,
         xreg=fourier(USAccDeaths, 5),
         seasonal=FALSE)

fc <- forecast(fit,
         xreg=fourierf(USAccDeaths, 5, 24))

plot(fc)
```

US Accidental Deaths



Outline

1 Regression with ARIMA errors

2 Stochastic and deterministic trends

3 Periodic seasonality

4 **Dynamic regression models**

Dynamic regression models

Sometimes a change in x_t does not affect y_t instantaneously

1. y_t = sales, x_t = advertising.
2. y_t = stream flow, x_t = rainfall.
3. y_t = size of herd, x_t = breeding stock.

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Lagged explanatory variables

The model include present and past values of predictor: $x_t, x_{t-1}, x_{t-2}, \ldots$.

$$y_t = a + \nu_0 x_t + \nu_1 x_{t-1} + \cdots + \nu_k x_{t-k} + n_t$$

where n_t is an ARIMA process.

Rewrite model as

$$\begin{aligned} y_t &= a + (\nu_0 + \nu_1 B + \nu_2 B^2 + \cdots + \nu_k B^k) x_t + n_t \\ &= a + \nu(B) x_t + n_t. \end{aligned}$$

- $\nu(B)$ is called a *transfer function* since it describes how change in x_t is transferred to y_t.
- x can influence y, but y is not allowed to influence x.

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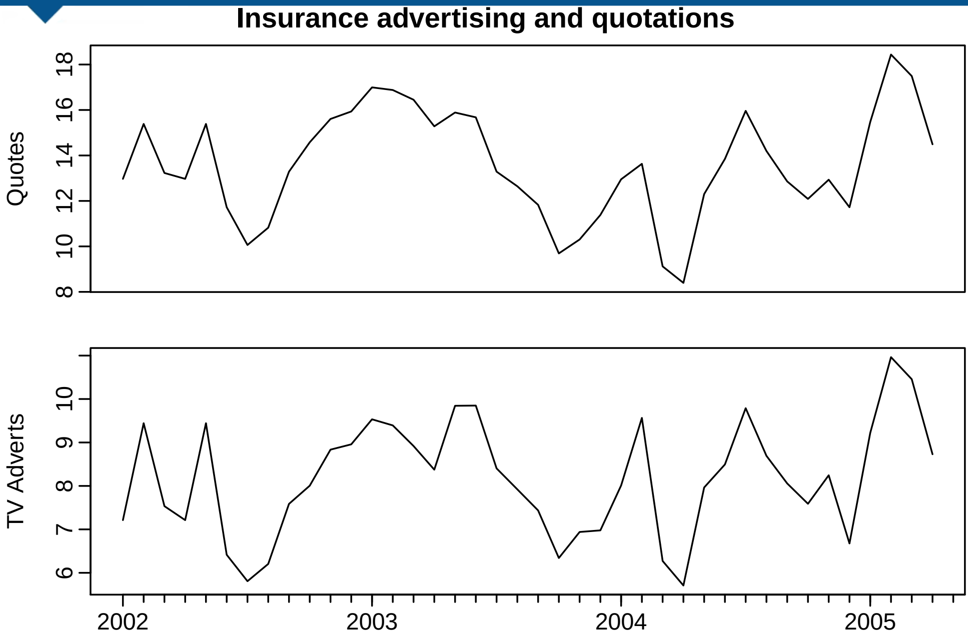
Example: Insurance quotes and TV adverts



Example: Insurance quotes and TV adverts

```
> Advert <- cbind(insurance[,2],
   c(NA,insurance[1:39,2]))
> colnames(Advert) <- c("AdLag0","AdLag1")
> fit <- auto.arima(insurance[,1], xreg=Advert, d=0)
ARIMA(3,0,0) with non-zero mean

Coefficients:
        ar1      ar2     ar3  intercept  AdLag0  AdLag1
     1.4117  -0.9317  0.3591     2.0393  1.2564  0.1625
s.e. 0.1698   0.2545  0.1592     0.9931  0.0667  0.0591

sigma^2 estimated as 0.1887:  log likelihood=-23.89
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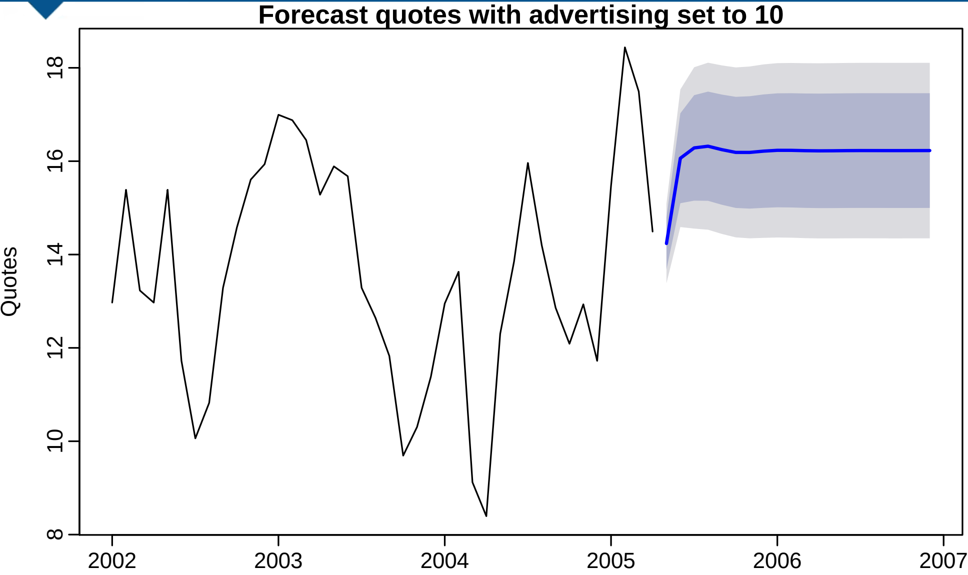

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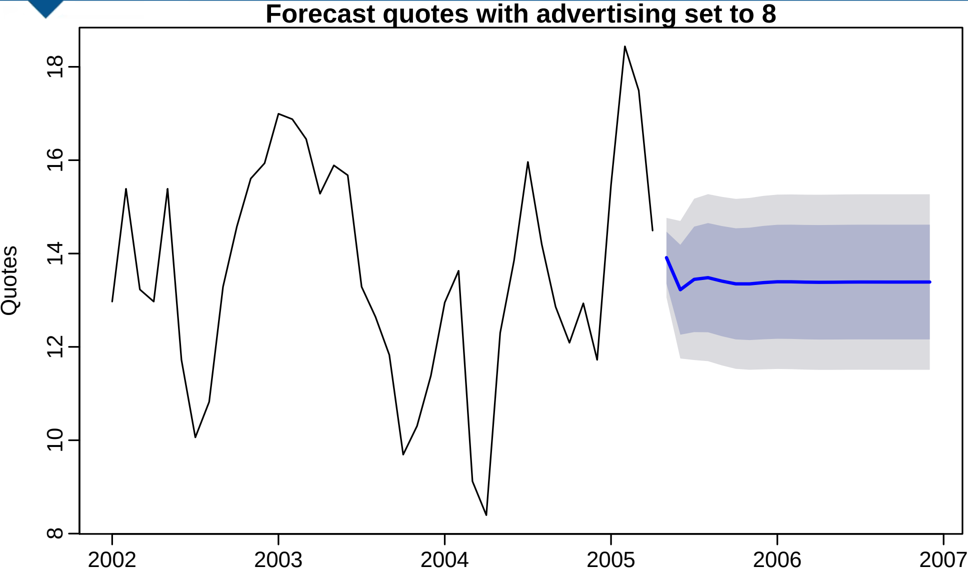

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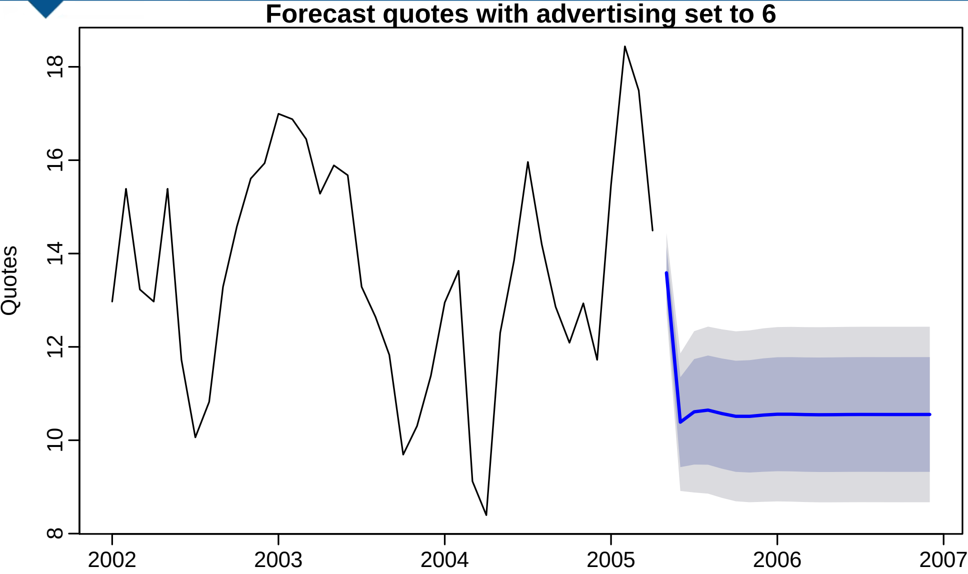

Example: Insurance quotes and TV adverts

```
fc <- forecast(fit, h=20,
  xreg=cbind(c(Advert[40,1],rep(6,19)), rep(6,20)))
plot(fc)
```

Dynamic regression models

$$y_t = a + \nu(B)x_t + n_t$$

where n_t is an ARMA process. So

$$\phi(B)n_t = \theta(B)e_t \qquad \text{or} \qquad n_t = \frac{\theta(B)}{\phi(B)}e_t = \psi(B)e_t.$$

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- ARMA models are rational approximations to general transfer functions of e_t.
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