MONASH University

REVOLUTION ANALYTICS

Rob J Hyndman

Forecasting using R

10. Seasonal ARIMA models

OTexts.com/fpp/8/9

Outline

1. **Backshift notation**
2. Seasonal ARIMA models
3. Example 1: European quarterly retail trade
4. Example 2: Australian cortecosteroid drug sales
5. ARIMA vs ETS

Backshift notation

A very useful notational device is the backward shift operator, B, which is used as follows:

$$By_t = y_{t-1} \,.$$

In other words, B, operating on y_t, has the effect of **shifting the data back one period**. Two applications of B to y_t **shifts the data back two periods**:

$$B(By_t) = B^2 y_t = y_{t-2} \,.$$

For monthly data, if we wish to shift attention to "the same month last year," then B^{12} is used, and the notation is $B^{12}y_t = y_{t-12}$.

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Backshift notation

- First difference: $1 - B$.
- Double difference: $(1 - B)^2$.
- dth-order difference: $(1 - B)^d y_t$.
- Seasonal difference: $1 - B^m$.
- Seasonal difference followed by a first difference: $(1 - B)(1 - B^m)$.
- Multiply terms together together to see the combined effect:

$$\begin{aligned}(1 - B)(1 - B^m)y_t &= (1 - B - B^m + B^{m+1})y_t \\ &= y_t - y_{t-1} - y_{t-m} + y_{t-m-1}.\end{aligned}$$

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Backshift notation for ARIMA

- ARMA model:

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + e_t + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q}$$
$$= c + \phi_1 B y_t + \cdots + \phi_p B^p y_t + e_t + \theta_1 B e_t + \cdots + \theta_q B^q e_t$$

$$\phi(B) y_t = c + \theta(B) e_t$$

where $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$

and $\theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q$.

- ARIMA(1,1,1) model:

$$(1 - \phi_1 B) \; (1 - B) y_t \; = \; c + (1 + \theta_1 B) e_t$$

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First difference

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AR(1)

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MA(1)

Outline

Seasonal ARIMA models

$$\text{ARIMA} \quad (p, d, q) \quad (P, D, Q)_m$$

where $m =$ number of periods per season.

Seasonal ARIMA models

$$\text{ARIMA} \quad \underbrace{(p, d, q)}_{\uparrow} \quad (P, D, Q)_m$$

$$\left(\begin{array}{l} \text{Non-} \\ \text{seasonal} \\ \text{part of the} \\ \text{model} \end{array} \right)$$

where $m =$ number of periods per season.

Seasonal ARIMA models

$$\text{ARIMA}\ \ (p,d,q)\ \ \underbrace{(P,D,Q)_m}_{\uparrow \atop \left(\begin{array}{l}\text{Seasonal}\\ \text{part of}\\ \text{the}\\ \text{model}\end{array}\right)}$$

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Seasonal ARIMA models

E.g., ARIMA$(1,1,1)(1,1,1)_4$ model (without constant)

$$(1-\phi_1 B)(1-\Phi_1 B^4)(1-B)(1-B^4)y_t = (1+\theta_1 B)(1+\Theta_1 B^4)e_t.$$

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(Seasonal difference)

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$\begin{pmatrix}\text{Non-seasonal}\\ \text{difference}\end{pmatrix}$

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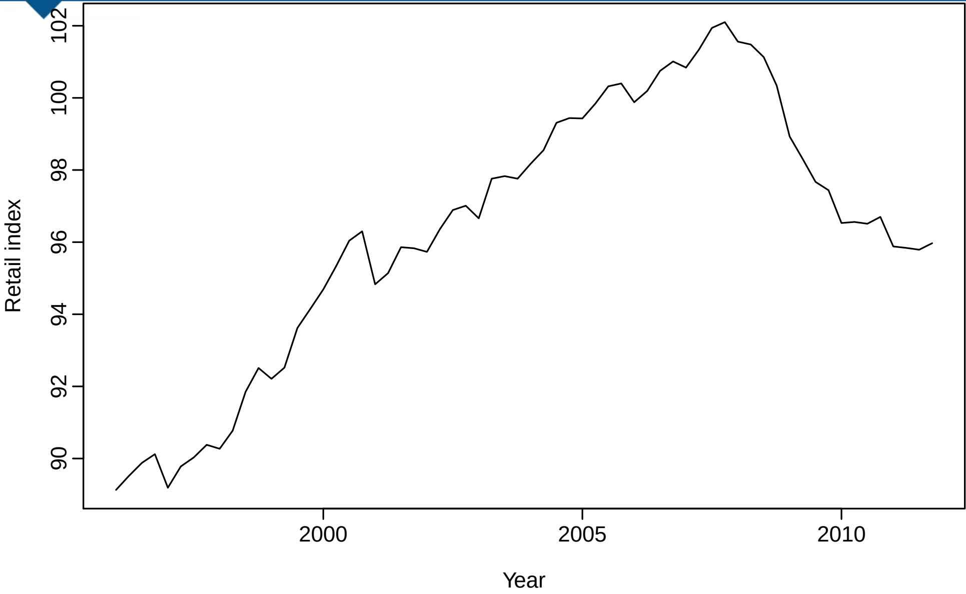
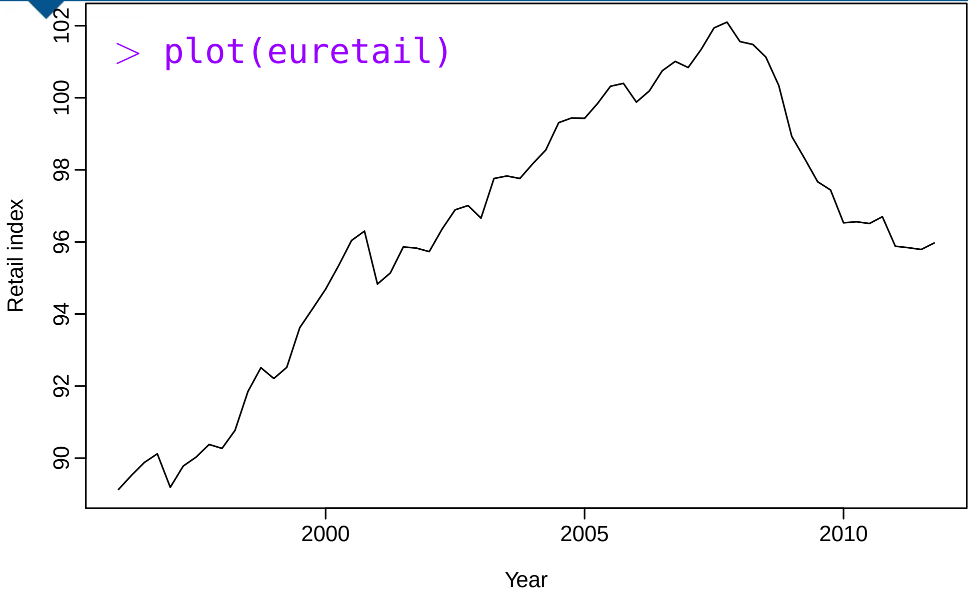
(Non-seasonal MA(1))

Outline

European quarterly retail trade



European quarterly retail trade



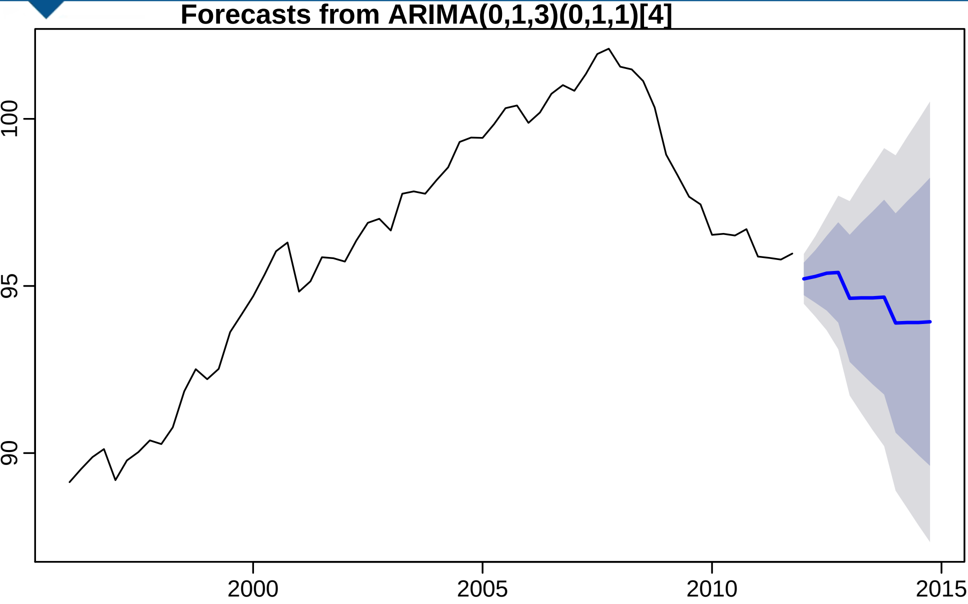
European quarterly retail trade

```
> auto.arima(euretail)
ARIMA(1,1,1)(0,1,1)[4]

Coefficients:
         ar1      ma1     sma1
      0.8828  -0.5208  -0.9704
s.e.  0.1424   0.1755   0.6792

sigma^2 estimated as 0.1411:  log likelihood=-30.19
AIC=68.37   AICc=69.11   BIC=76.68
```

European quarterly retail trade

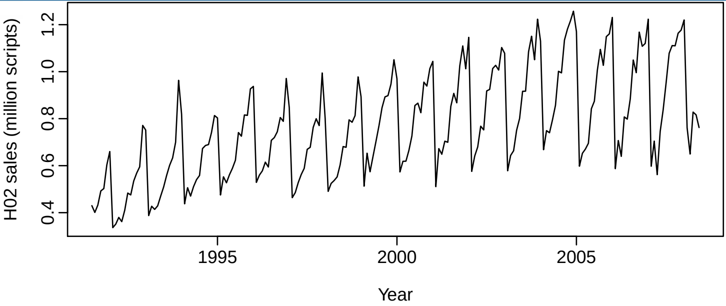
```
> auto.arima(euretail, stepwise=FALSE,
    approximation=FALSE)
ARIMA(0,1,3)(0,1,1)[4]

Coefficients:
         ma1     ma2     ma3     sma1
      0.2625  0.3697  0.4194  -0.6615
s.e.  0.1239  0.1260  0.1296   0.1555

sigma^2 estimated as 0.1451:  log likelihood=-28.7
AIC=67.4   AICc=68.53   BIC=77.78
```

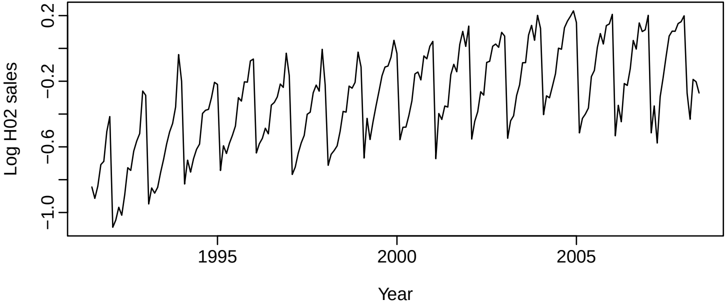
European quarterly retail trade



Outline

Cortecosteroid drug sales

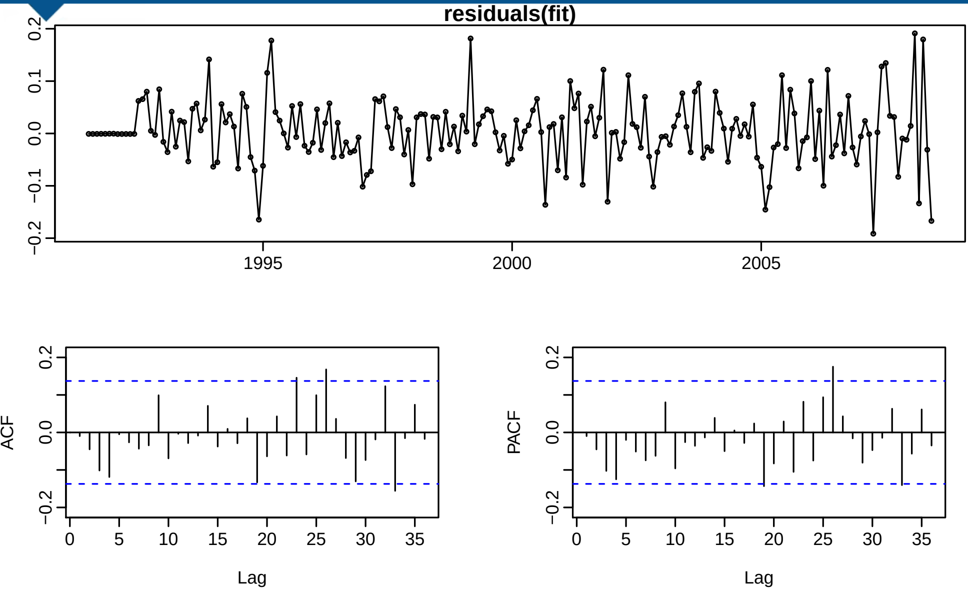




Cortecosteroid drug sales

```
> fit <- auto.arima(h02, lambda=0)
> fit
ARIMA(2,1,3)(0,1,1)[12]
Box Cox transformation: lambda= 0

Coefficients:
          ar1      ar2     ma1     ma2      ma3     sma
      -1.0194  -0.8351  0.1717  0.2578  -0.4206  -0.652
s.e.   0.1648   0.1203  0.2079  0.1177   0.1060   0.065

sigma^2 estimated as 0.004071:  log likelihood=250.8
AIC=-487.6   AICc=-486.99   BIC=-464.83
```

Cortecosteroid drug sales



Cortecosteroid drug sales

Training: July 91 – June 06

Test: July 06 – June 08

Model	RMSE
ARIMA$(3,0,0)(2,1,0)_{12}$	0.0661
ARIMA$(3,0,1)(2,1,0)_{12}$	0.0646
ARIMA$(3,0,2)(2,1,0)_{12}$	0.0645
ARIMA$(3,0,1)(1,1,0)_{12}$	0.0679
ARIMA$(3,0,1)(0,1,1)_{12}$	0.0644
ARIMA$(3,0,1)(0,1,2)_{12}$	0.0622
ARIMA$(3,0,1)(1,1,1)_{12}$	0.0630
ARIMA$(4,0,3)(0,1,1)_{12}$	0.0648
ARIMA$(3,0,3)(0,1,1)_{12}$	0.0640
ARIMA$(4,0,2)(0,1,1)_{12}$	0.0648
ARIMA$(3,0,2)(0,1,1)_{12}$	0.0644
ARIMA$(2,1,3)(0,1,1)_{12}$	0.0634
ARIMA$(2,1,4)(0,1,1)_{12}$	0.0632
ARIMA$(2,1,5)(0,1,1)_{12}$	0.0640

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Cortecosteroid drug sales

```
getrmse <- function(x,h,...)
{
  train.end <- time(x)[length(x)-h]
  test.start <- time(x)[length(x)-h+1]
  train <- window(x,end=train.end)
  test <- window(x,start=test.start)
  fit <- Arima(train,...)
  fc <- forecast(fit,h=h)
  return(accuracy(fc,test)["RMSE"])
}
```

Cortecosteroid drug sales

```
getrmse(h02,h=24,order=c(3,0,0),seasonal=c(2,1,0),lambda=0)
getrmse(h02,h=24,order=c(3,0,1),seasonal=c(2,1,0),lambda=0)
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getrmse(h02,h=24,order=c(2,1,3),seasonal=c(0,1,1),lambda=0)
getrmse(h02,h=24,order=c(2,1,4),seasonal=c(0,1,1),lambda=0)
getrmse(h02,h=24,order=c(2,1,5),seasonal=c(0,1,1),lambda=0)
```

Cortecosteroid drug sales

- Models with lowest AIC_c values tend to give slightly better results than the other models.
- AIC_c comparisons must have the same orders of differencing. But RMSE test set comparisons can involve any models.
- No model passes all the residual tests.
- Use the best model available, even if it does not pass all tests.

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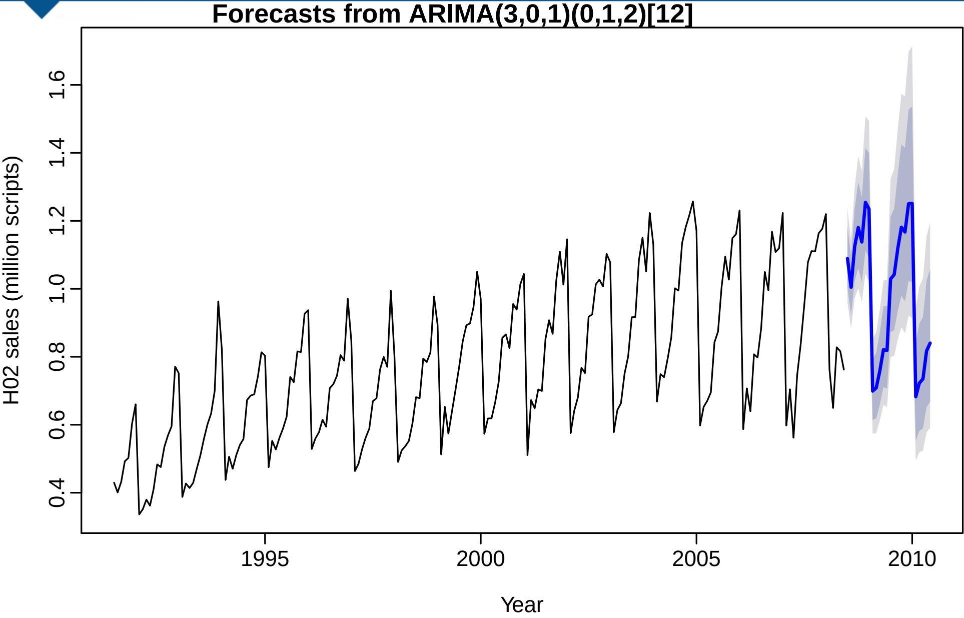

Outline

ARIMA vs ETS

- Myth that ARIMA models are more general than exponential smoothing.
- Linear exponential smoothing models all special cases of ARIMA models.
- Non-linear exponential smoothing models have no equivalent ARIMA counterparts.
- Many ARIMA models have no exponential smoothing counterparts.
- ETS models all non-stationary. Models with seasonality or non-damped trend (or both) have two unit roots; all other models have one unit root.

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Equivalences

Simple exponential smoothing

- Forecasts equivalent to **ARIMA(0,1,1)**.
- Parameters: $\theta_1 = \alpha - 1$.

Holt's method

- Forecasts equivalent to **ARIMA(0,2,2)**.
- Parameters: $\theta_1 = \alpha + \beta - 2$ and $\theta_2 = 1 - \alpha$.

Damped Holt's method

- Forecasts equivalent to **ARIMA(1,1,2)**.
- Parameters: $\phi_1 = \phi$, $\theta_1 = \alpha + \phi\beta - 2$, $\theta_2 = (1 - \alpha)\phi$.

Holt-Winters' additive method

- Forecasts equivalent to **ARIMA(0,1,m+1)(0,1,0)$_m$**.
- Parameter restrictions because ARIMA has $m + 1$ parameters whereas HW uses only three parameters.

Holt-Winters' multiplicative method

- No ARIMA equivalence

Equivalences

Simple exponential smoothing

- Forecasts equivalent to **ARIMA(0,1,1)**.
- Parameters: $\theta_1 = \alpha - 1$.

Holt's method

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Damped Holt's method

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Holt-Winters' additive method

- Forecasts equivalent to **ARIMA(0,1,m+1)(0,1,0)$_m$**.
- Parameter restrictions because ARIMA has $m + 1$ parameters whereas HW uses only three parameters.

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- No ARIMA equivalence

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