10. Seasonal ARIMA models

OTexts.com/fpp/8/9
Outline

1 Backshift notation
2 Seasonal ARIMA models
3 Example 1: European quarterly retail trade
4 Example 2: Australian cortecosteroid drug sales
5 ARIMA vs ETS
A very useful notational device is the backward shift operator, $B$, which is used as follows:

$$By_t = y_{t-1}.$$ 

In other words, $B$, operating on $y_t$, has the effect of shifting the data back one period. Two applications of $B$ to $y_t$ shifts the data back two periods:

$$B(By_t) = B^2 y_t = y_{t-2}.$$ 

For monthly data, if we wish to shift attention to “the same month last year,” then $B^{12}$ is used, and the notation is $B^{12} y_t = y_{t-12}$.
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Backshift notation

- **First difference:** $1 - B$.
- **Double difference:** $(1 - B)^2$.
- **$d$th-order difference:** $(1 - B)^d y_t$.
- **Seasonal difference:** $1 - B^m$.
- **Seasonal difference followed by a first difference:** $(1 - B)(1 - B^m)$.
- **Multiply terms together to see the combined effect:**

  $$(1 - B)(1 - B^m)y_t = (1 - B - B^m + B^{m+1})y_t$$

  $$= y_t - y_{t-1} - y_{t-m} + y_{t-m-1}.$$
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Backshift notation for ARIMA

- **ARMA model:**
  
  \[
  y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + e_t + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q}
  \]
  
  \[
  = c + \phi_1 B y_t + \cdots + \phi_p B^p y_t + e_t + \theta_1 B e_t + \cdots + \theta_q B^q e_t
  \]
  
  \[
  \phi(B)y_t = c + \theta(B)e_t
  \]

  where \( \phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p \)

  and \( \theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q \).

- **ARIMA(1,1,1) model:**

  \[
  (1 - \phi_1 B) \quad (1 - B)y_t = c + (1 + \theta_1 B)e_t
  \]
Backshift notation for ARIMA

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  \[ y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + e_t + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q} \]
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- **ARIMA(1,1,1) model:**
  \[ (1 - \phi_1 B) (1 - B)y_t = c + (1 + \theta_1 B)e_t \]
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  ↑

  First difference
Backshift notation for ARIMA

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\[ = c + \phi_1 B y_t + \cdots + \phi_p B^p y_t + e_t + \theta_1 B e_t + \cdots + \theta_q B^q e_t \]

\[ \phi(B) y_t = c + \theta(B) e_t \]

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\[ \uparrow \]

AR(1)
Backshift notation for ARIMA

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Forecasting using R

Seasonal ARIMA models

\[ \text{ARIMA} (p, d, q) \ (P, D, Q)_m \]

where \( m = \) number of periods per season.
Seasonal ARIMA models

ARIMA \((p, d, q) \quad (P, D, Q)_m\)

\[
\begin{pmatrix}
\text{Non-seasonal part of the model}
\end{pmatrix}
\]

where \(m = \text{number of periods per season.}\)
Seasonal ARIMA models

ARIMA \((p, d, q)\) \((P, D, Q)_m\)

\[\text{Seasonal part of the model}\]

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Seasonal ARIMA models

E.g., ARIMA(1, 1, 1)(1, 1, 1)_4 model (without constant)

\[(1 - \phi_1 B)(1 - \Phi_1 B^4)(1 - B)(1 - B^4)y_t = (1 + \theta_1 B)(1 + \Theta_1 B^4)e_t.\]
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(Non-seasonal difference)

Forecasting using R

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Outline

1. Backshift notation
2. Seasonal ARIMA models
3. Example 1: European quarterly retail trade
4. Example 2: Australian cortecosteroid drug sales
5. ARIMA vs ETS
European quarterly retail trade forecasting using R

Example 1: European quarterly retail trade

<table>
<thead>
<tr>
<th>Year</th>
<th>Retail index</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>90</td>
</tr>
<tr>
<td>2005</td>
<td>92</td>
</tr>
<tr>
<td>2010</td>
<td>94</td>
</tr>
</tbody>
</table>
European quarterly retail trade

> plot(euretail)
> auto.arima(euretail)
ARIMA(1,1,1)(0,1,1)[4]

Coefficients:

<table>
<thead>
<tr>
<th>ar1</th>
<th>ma1</th>
<th>sma1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8828</td>
<td>-0.5208</td>
<td>-0.9704</td>
</tr>
</tbody>
</table>

s.e. 0.1424 0.1755 0.6792

sigma^2 estimated as 0.1411: log likelihood=-30.19
AIC=68.37  AICc=69.11  BIC=76.68
> auto.arima(euretail, stepwise=FALSE, approximation=FALSE)
ARIMA(0,1,3)(0,1,1)[4]

Coefficients:

\[
\begin{array}{ccccc}
\text{ma1} & \text{ma2} & \text{ma3} & \text{sma1} \\
0.2625 & 0.3697 & 0.4194 & -0.6615 \\
\text{s.e.} & 0.1239 & 0.1260 & 0.1296 & 0.1555 \\
\end{array}
\]

\(\sigma^2\) estimated as 0.1451: log likelihood=-28.7
AIC=67.4 AICc=68.53 BIC=77.78
Example 1: European quarterly retail trade

Forecasts from ARIMA(0,1,3)(0,1,1)[4]
Outline

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Cortecosteroid drug sales

Example 2: Australian cortecosteroid drug sales

Year
H02 sales (million scripts)
1995 2000 2005
0.4 0.6 0.8 1.0 1.2
Year
Log H02 sales
1995 2000 2005
−1.0 −0.6 −0.2 0.2

Forecasting using R
Cortecosteroid drug sales

> fit <- auto.arima(h02, lambda=0)
> fit
ARIMA(2,1,3)(0,1,1)[12]
Box Cox transformation: lambda= 0

Coefficients:

<table>
<thead>
<tr>
<th></th>
<th>ar1</th>
<th>ar2</th>
<th>ma1</th>
<th>ma2</th>
<th>ma3</th>
<th>sma1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-1.0194</td>
<td>-0.8351</td>
<td>0.1717</td>
<td>0.2578</td>
<td>-0.4206</td>
<td>-0.6528</td>
</tr>
</tbody>
</table>

s.e. 0.1648 0.1203 0.2079 0.1177 0.1060 0.0657

sigma^2 estimated as 0.004071: log likelihood=250.8
AIC=-487.6  AICc=-486.99  BIC=-464.83
Cortecosteroid drug sales

Example 2: Australian cortecosteroid drug sales

residuals(fit)
# Corticosteroid drug sales

**Training:** July 91 – June 06

**Test:** July 06 – June 08

<table>
<thead>
<tr>
<th>Model</th>
<th>RMSE</th>
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<tr>
<td>ARIMA(3,0,0)(2,1,0)_{12}</td>
<td>0.0661</td>
</tr>
<tr>
<td>ARIMA(3,0,1)(2,1,0)_{12}</td>
<td>0.0646</td>
</tr>
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<td>ARIMA(3,0,2)(2,1,0)_{12}</td>
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<tr>
<td>ARIMA(2,1,4)(0,1,1)_{12}</td>
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<tr>
<td>ARIMA(2,1,5)(0,1,1)_{12}</td>
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Cortecosteroid drug sales

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getrmse <- function(x,h,...)
{
  train.end <- time(x)[length(x)-h]
  test.start <- time(x)[length(x)-h+1]
  train <- window(x,end=train.end)
  test <- window(x,start=test.start)
  fit <- Arima(train,...)
  fc <- forecast(fit,h=h)
  return(accuracy(fc,test)["RMSE"])
}
Cortecosteroid drug sales

getrmse(h02,h=24,order=c(3,0,0),seasonal=c(2,1,0),lambda=0)
getrmse(h02,h=24,order=c(3,0,1),seasonal=c(2,1,0),lambda=0)
getrmse(h02,h=24,order=c(3,0,2),seasonal=c(2,1,0),lambda=0)
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getrmse(h02,h=24,order=c(3,0,1),seasonal=c(1,1,1),lambda=0)
getrmse(h02,h=24,order=c(4,0,3),seasonal=c(0,1,1),lambda=0)
getrmse(h02,h=24,order=c(3,0,3),seasonal=c(0,1,1),lambda=0)
getrmse(h02,h=24,order=c(4,0,2),seasonal=c(0,1,1),lambda=0)
getrmse(h02,h=24,order=c(3,0,2),seasonal=c(0,1,1),lambda=0)
getrmse(h02,h=24,order=c(2,1,3),seasonal=c(0,1,1),lambda=0)
getrmse(h02,h=24,order=c(2,1,4),seasonal=c(0,1,1),lambda=0)
getrmse(h02,h=24,order=c(2,1,5),seasonal=c(0,1,1),lambda=0)

Forecasting using R

Example 2: Australian cortecosteroid drug sales
Models with lowest $\text{AIC}_c$ values tend to give slightly better results than the other models.

$\text{AIC}_c$ comparisons must have the same orders of differencing. But RMSE test set comparisons can involve any models.

No model passes all the residual tests.

Use the best model available, even if it does not pass all tests.
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Cortecosteroid drug sales

Forecasts from ARIMA(3,0,1)(0,1,2)[12]

Year
H02 sales (million scripts)
1995 2000 2005 2010
0.4 0.6 0.8 1.0 1.2 1.4 1.6

Year
1995 2000 2005 2010
1. Backshift notation

2. Seasonal ARIMA models

3. Example 1: European quarterly retail trade

4. Example 2: Australian cortecosteroid drug sales

5. ARIMA vs ETS
Myth that ARIMA models are more general than exponential smoothing.

- Linear exponential smoothing models all special cases of ARIMA models.
- Non-linear exponential smoothing models have no equivalent ARIMA counterparts.
- Many ARIMA models have no exponential smoothing counterparts.
- ETS models all non-stationary. Models with seasonality or non-damped trend (or both) have two unit roots; all other models have one unit root.
ARIMA vs ETS

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- Linear exponential smoothing models all special cases of ARIMA models.
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Equivalences

Simple exponential smoothing

- Forecasts equivalent to $\text{ARIMA}(0,1,1)$.
- Parameters: $\theta_1 = \alpha - 1$.

Holt’s method

- Forecasts equivalent to $\text{ARIMA}(0,2,2)$.
- Parameters: $\theta_1 = \alpha + \beta - 2$ and $\theta_2 = 1 - \alpha$.

Damped Holt’s method

- Forecasts equivalent to $\text{ARIMA}(1,1,2)$.
- Parameters: $\phi_1 = \phi$, $\theta_1 = \alpha + \phi \beta - 2$, $\theta_2 = (1 - \alpha)\phi$.

Holt-Winters’ additive method

- Forecasts equivalent to $\text{ARIMA}(0,1,m+1)(0,1,0)_m$.
- Parameter restrictions because ARIMA has $m+1$ parameters whereas HW uses only three parameters.

Holt-Winters’ multiplicative method

- No ARIMA equivalence.
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