MONASH University

Rob J Hyndman

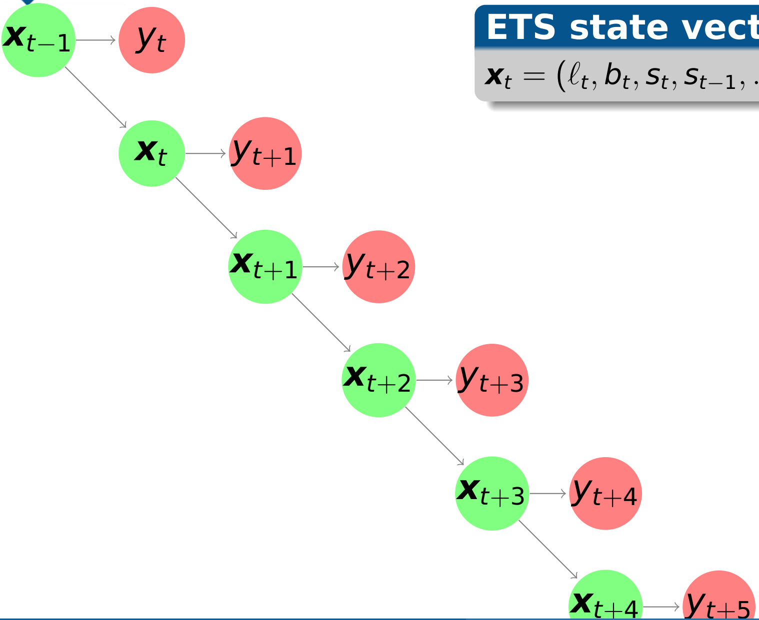
State space models

2: Structural models

Outline

Outline

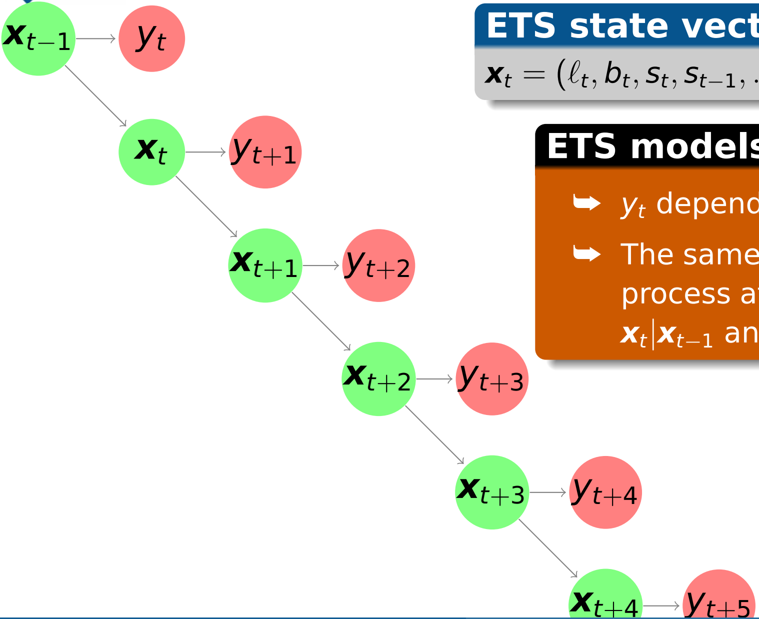
State space models



ETS state vector

$$\boldsymbol{x}_t = (\ell_t, b_t, s_t, s_{t-1}, \dots, s_{t-m+1})$$

State space models



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- y_t depends on $\boldsymbol{x}_{t-1}$.
- The same error process affects $\boldsymbol{x}_t|\boldsymbol{x}_{t-1}$ and $y_t|\boldsymbol{x}_{t-1}$.

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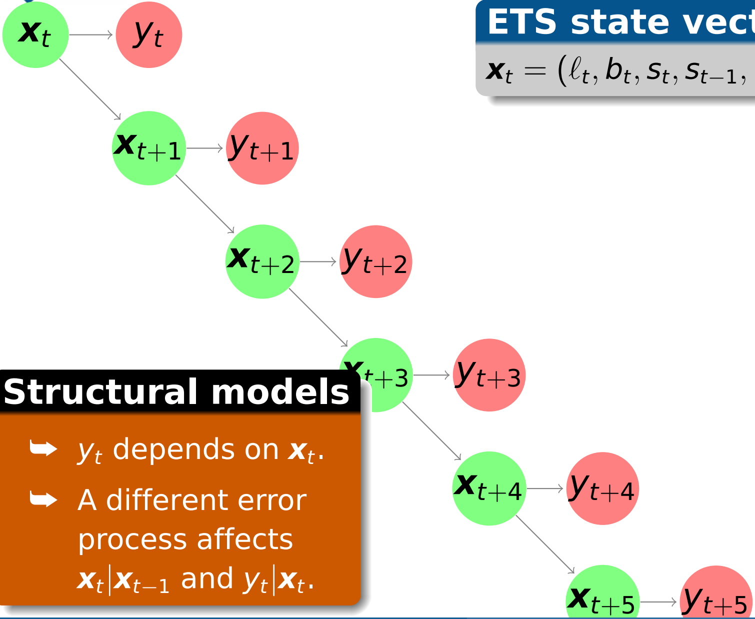

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Structural models

- y_t depends on $\boldsymbol{x}_t$.
- A different error process affects $\boldsymbol{x}_t|\boldsymbol{x}_{t-1}$ and $y_t|\boldsymbol{x}_t$.

Local level model

Stochastically varying level (random walk) observed with noise

$$y_t = \ell_t + \varepsilon_t$$
$$\ell_t = \ell_{t-1} + \xi_t$$

- ε_t and ξ_t are independent Gaussian white noise processes.
- Compare ETS(A,N,N) where $\xi_t = \alpha\varepsilon_{t-1}$.
- Parameters to estimate: σ_ε^2 and σ_ξ^2.
- If $\sigma_\xi^2 = 0$, $y_t \sim \text{NID}(\ell_0, \sigma_\varepsilon^2)$.

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Local linear trend model

Dynamic trend observed with noise

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Basic structural model

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y_t &= \ell_t + \sum_{j=1}^{J} s_{j,t} + \varepsilon_t \\
\ell_t &= \ell_{t-1} + b_{t-1} + \xi_t \\
b_t &= b_{t-1} + \zeta_t \\
s_{j,t} &= \quad \cos\lambda_j s_{j,t-1} + \sin\lambda_j s^*_{j,t-1} + \omega_{j,t} \\
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- $\lambda_j = 2\pi j/m$
- ε_t, ξ_t, ζ_t, $\omega_{j,t}$, $\omega^*_{j,t}$ are independent Gaussian white noise processes
- $\omega_{j,t}$ and $\omega^*_{j,t}$ have same variance $\sigma^2_{\omega,j}$
- Equivalent to BSM when $\sigma^2_{\omega,j} = \sigma^2_{\omega}$ and $J = m/2$
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ETS vs Structural models

- ETS models are much more general as they allow non-linear (multiplicative components).
- ETS allows automatic forecasting due to its larger model space.
- Additive ETS models are almost equivalent to the corresponding structural models.
- ETS models have a larger parameter space. Structural models parameters are always non-negative (variances).
- Structural models are much easier to generalize (e.g., add covariates).
- It is easier to handle missing values with structural models.

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Structural models in R

```
StructTS(oil, type="level")
StructTS(ausair, type="trend")
StructTS(austourists, type="BSM")

fit <- StructTS(austourists, type = "BSM")
decomp <- cbind(austourists, fitted(fit))
colnames(decomp) <- c("data","level","slope",
   "seasonal")
plot(decomp, main="Decomposition of
  International visitor nights")
```

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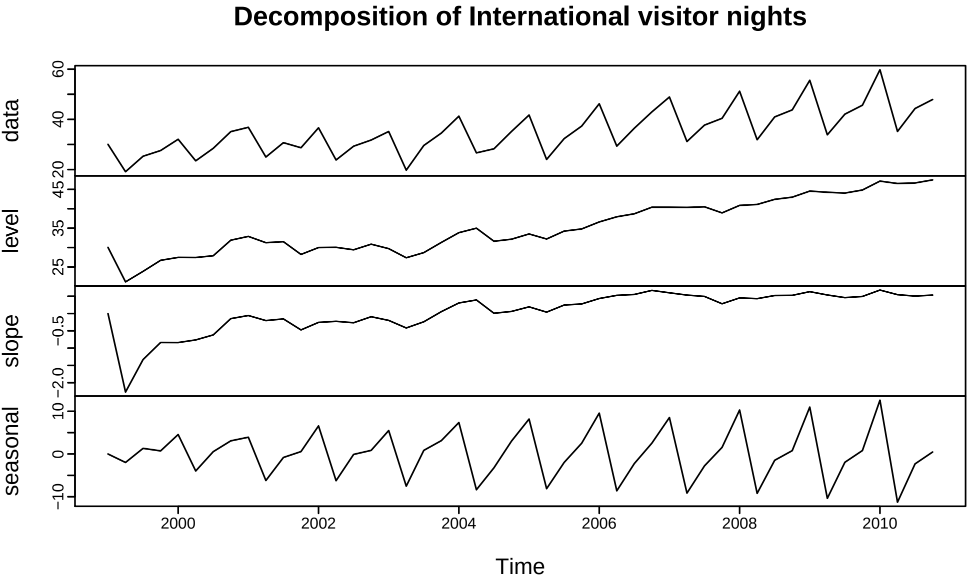
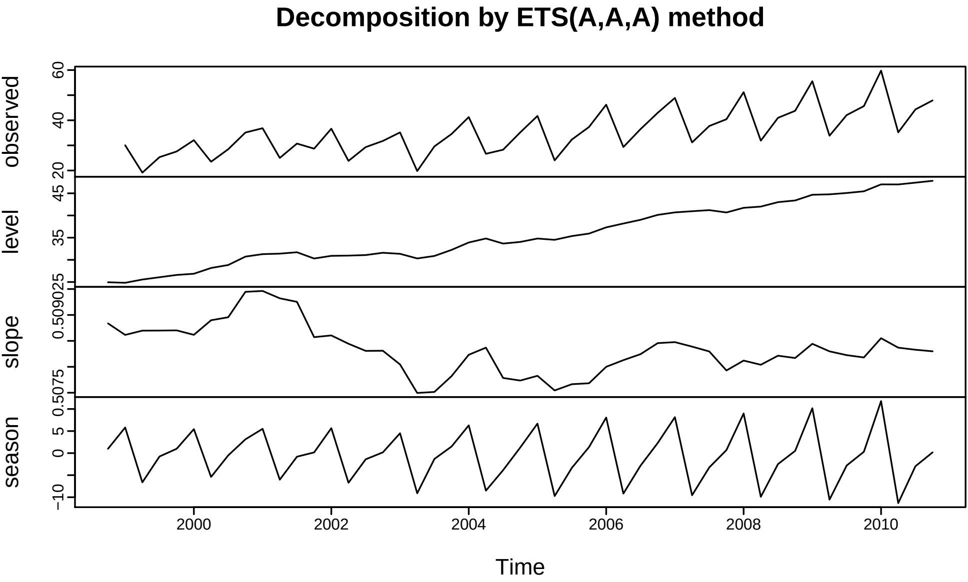


ETS decomposition



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Linear Gaussian SS models

Observation equation	$y_t = \boldsymbol{f}'\boldsymbol{x}_t + \varepsilon_t$
State equation	$\boldsymbol{x}_t = \boldsymbol{G}\boldsymbol{x}_{t-1} + \boldsymbol{w}_t$

- State vector $\boldsymbol{x}_t$ of length p
- $\boldsymbol{G}$ a $p \times p$ matrix, $\boldsymbol{f}$ a vector of length p
- $\varepsilon_t \sim \text{NID}(0, \sigma^2)$, $\quad \boldsymbol{w}_t \sim \text{NID}(0, \boldsymbol{W})$.

Local level model:

$\boldsymbol{f} = \boldsymbol{G} = 1, \qquad \boldsymbol{x}_t = \ell_t.$

Local linear trend model:

$\boldsymbol{f}' = [1\ 0],$

$$\boldsymbol{x}_t = \begin{bmatrix} \ell_t \\ b_t \end{bmatrix} \qquad \boldsymbol{G} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \boldsymbol{W} = \begin{bmatrix} \sigma_\xi^2 & 0 \\ 0 & \sigma_\zeta^2 \end{bmatrix}$$

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Linear Gaussian state space model

$$y_t = \boldsymbol{f}'\boldsymbol{x}_t + \varepsilon_t, \qquad \varepsilon_t \sim \mathrm{N}(0, \sigma^2)$$
$$\boldsymbol{x}_t = \boldsymbol{G}\boldsymbol{x}_{t-1} + \boldsymbol{w}_t \qquad \boldsymbol{w}_t \sim \mathrm{N}(\boldsymbol{0}, \boldsymbol{W})$$

$$\boldsymbol{f}' = [1\ 0\ 1\ 0\ \cdots\ 0], \quad \boldsymbol{W} = \mathrm{diagonal}(\sigma_\xi^2, \sigma_\zeta^2, \sigma_\eta^2, 0, \dots, 0)$$

$$\boldsymbol{x}_t = \begin{bmatrix} \ell_t \\ b_t \\ s_{1,t} \\ s_{2,t} \\ s_{3,t} \\ \vdots \\ s_{m-1,t} \end{bmatrix} \quad \boldsymbol{G} = \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & -1 & \dots & -1 & -1 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{bmatrix}$$

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Outline

Kalman filter

Notation:

$$\hat{\boldsymbol{x}}_{t|t} = \mathrm{E}[\boldsymbol{x}_t | y_1, \dots, y_t] \qquad \hat{\boldsymbol{P}}_{t|t} = \mathrm{V}[\boldsymbol{x}_t | y_1, \dots, y_t]$$

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Updating or State Filtering:

$$\hat{\boldsymbol{x}}_{t|t} = \hat{\boldsymbol{x}}_{t|t-1} + \hat{\boldsymbol{P}}_{t|t-1}\boldsymbol{f}\hat{v}_{t|t-1}^{-1}(y_t - \hat{y}_{t|t-1})$$

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$$\hat{\boldsymbol{x}}_{t+1|t} = \boldsymbol{G}\hat{\boldsymbol{x}}_{t|t}$$

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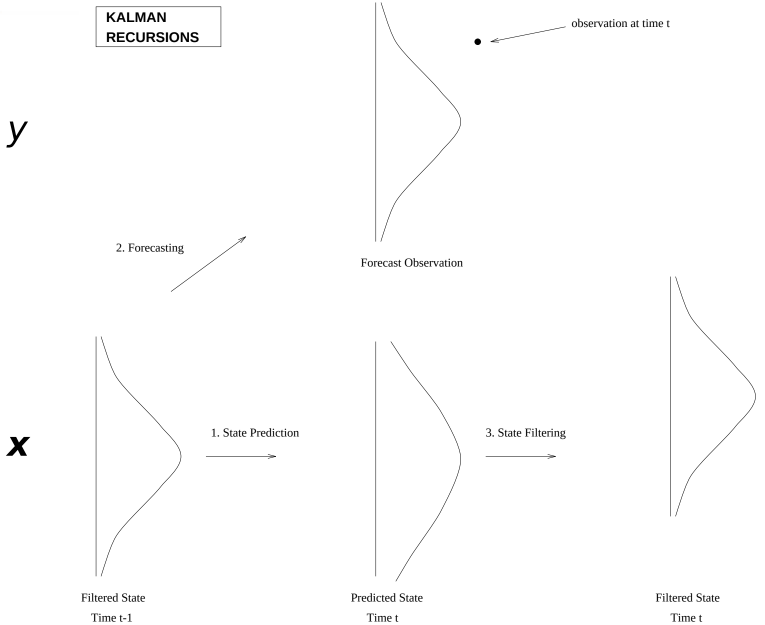
Just conditional expectations. So this gives minimum MSE estimates.

Kalman recursions



Initializing Kalman filter

- Need $\boldsymbol{x}_{1|0}$ and $\boldsymbol{P}_{1|0}$ to get started.
- Common approach for structural models: set $\boldsymbol{x}_{1|0} = 0$ and $\boldsymbol{P}_{1|0} = k\boldsymbol{I}$ for a very large k.
- Lots of research papers on optimal initialization choices for Kalman recursions.
- ETS approach was to estimate $\boldsymbol{x}_{1|0}$ and avoid $\boldsymbol{P}_{1|0}$ by assuming error processes identical.
- A random $\boldsymbol{x}_{1|0}$ could be used with ETS models, and then a form of Kalman filter would be required for estimation and forecasting.
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Local level model

$$y_t = \ell_t + \varepsilon_t \qquad \varepsilon_t \sim \text{NID}(0, \sigma^2)$$
$$\ell_t = \ell_{t-1} + u_t \qquad u_t \sim \text{NID}(0, q^2)$$

Kalman recursions:

$$\hat{y}_{t|t-1} = \hat{\ell}_{t-1|t-1}$$
$$\hat{v}_{t|t-1} = \hat{p}_{t|t-1} + \sigma^2$$
$$\hat{\ell}_{t|t} = \hat{\ell}_{t-1|t-1} + \hat{p}_{t|t-1}\hat{v}^{-1}_{t|t-1}(y_t - \hat{y}_{t|t-1})$$
$$\hat{p}_{t+1|t} = \hat{p}_{t|t-1}(1 - \hat{v}^{-1}_{t|t-1}\hat{p}_{t|t-1}) + q^2$$

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\end{aligned}$$

Handling missing values

Forecasting:

$$\hat{y}_{t|t-1} = \boldsymbol{f}'\hat{\boldsymbol{x}}_{t|t-1}$$
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Iterate for $t = 1, \dots, T$ starting with $\boldsymbol{x}_{1|0}$ and $\boldsymbol{P}_{1|0}$.

Updating or State Filtering:

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State Prediction

$$\hat{\boldsymbol{x}}_{t|t-1} = \boldsymbol{G}\hat{\boldsymbol{x}}_{t-1|t-1}$$
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Multi-step forecasting

Forecasting:

$$\hat{y}_{t|t-1} = \boldsymbol{f}'\hat{\boldsymbol{x}}_{t|t-1}$$
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Iterate for $t = T+1, \dots, T+h$ starting with $\boldsymbol{x}_{T|T}$ and $\boldsymbol{P}_{T|T}$.

Updating or State Filtering:

$$\hat{\boldsymbol{x}}_{t|t} = \hat{\boldsymbol{x}}_{t|t-1} \cancel{+ \hat{\boldsymbol{P}}_{t|t-1}\boldsymbol{f}\hat{v}_{t|t-1}^{-1}(y_t - \hat{y}_{t|t-1})}$$
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Updating or State Filtering:

$$\hat{\boldsymbol{x}}_{t|t} = \hat{\boldsymbol{x}}_{t|t-1} \cancel{+ \hat{\boldsymbol{P}}_{t|t-1}\boldsymbol{f}\hat{v}_{t|t-1}^{-1}(y_t - \hat{y}_{t|t-1})}$$

$$\hat{\boldsymbol{P}}_{t|t} = \hat{\boldsymbol{P}}_{t|t-1} \cancel{- \hat{\boldsymbol{P}}_{t|t-1}\boldsymbol{f}\hat{v}_{t|t-1}^{-1}\boldsymbol{f}'\hat{\boldsymbol{P}}_{t|t-1}}$$

State Prediction

$$\hat{\boldsymbol{x}}_{t|t-1} = \boldsymbol{G}\hat{\boldsymbol{x}}_{t-1|t-1}$$

$$\hat{\boldsymbol{P}}_{t|t-1} = \boldsymbol{G}\hat{\boldsymbol{P}}_{t-1|t-1}\boldsymbol{G}' + \boldsymbol{W}$$

Treat future values as missing.

Kalman filter

What's so special about the Kalman filter

- Very general equations for any model in state space format.
- Any model in state space format can easily be generalized.
- Optimal MSE forecasts
- Easy to handle missing values.
- Easy to compute likelihood.

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Likelihood calculation

θ = all unknown parameters

$f_\theta(y_t|y_1, y_2, \ldots, y_{t-1})$ = one-step forecast density.

Likelihood

$$L(y_1, \ldots, y_T; \theta) = \prod_{t=1}^{T} f_\theta(y_t|y_1, \ldots, y_{t-1})$$

Gaussian log likelihood

$$\log L = -\frac{T}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\log \hat{v}_{t|t-1} - \frac{1}{2}\sum_{t=1}^{T} e_t^2/\hat{v}_{t|t-1}$$

where $e_t = y_t - \hat{y}_{t|t-1}$.

All terms obtained from Kalman filter equations.

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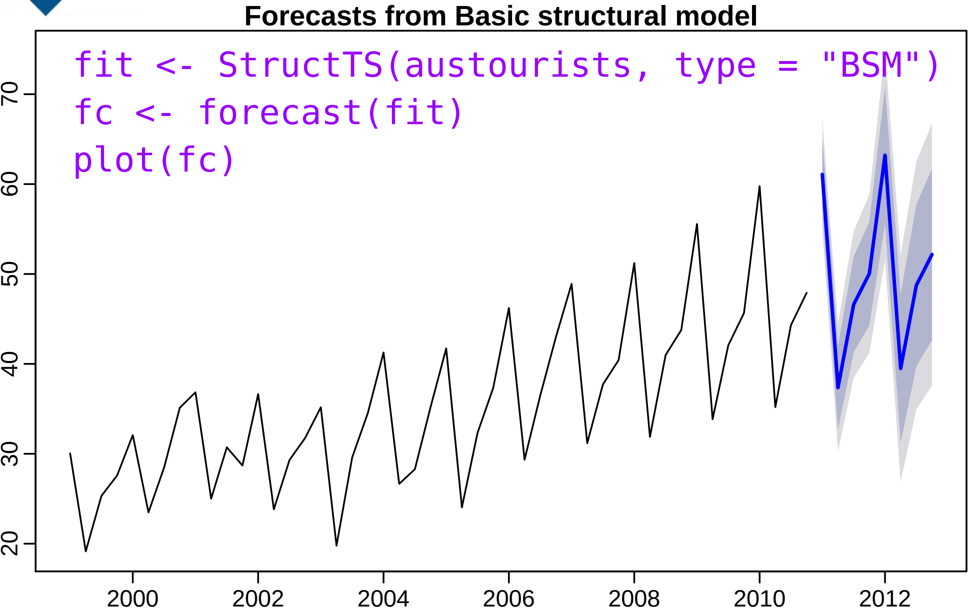
All terms obtained from Kalman filter equations.

Structural models in R

```
fit <- StructTS(austourists, type = "BSM")
fc <- forecast(fit)
plot(fc)
```



Outline

Kalman smoothing

Want estimate of $\boldsymbol{x}_t | y_1, \dots, y_T$ where $t < T$. That is, $\hat{\boldsymbol{x}}_{t|T}$.

$$
\begin{aligned}
\hat{\boldsymbol{x}}_{t|T} &= \hat{\boldsymbol{x}}_{t|t} + \boldsymbol{A}_t \left(\hat{\boldsymbol{x}}_{t+1|T} - \hat{\boldsymbol{x}}_{t+1|t} \right) \\
\hat{P}_{t|T} &= \hat{P}_{t|t} + \boldsymbol{A}_t \left(\hat{P}_{t+1|T} - \hat{P}_{t+1|t} \right) \boldsymbol{A}_t' \\
\text{where} \quad A_t &= \hat{P}_{t|t} \boldsymbol{G}' \left(\hat{P}_{t+1|t} \right)^{-1}.
\end{aligned}
$$

- Uses all data, not just previous data.
- Useful for estimating missing values: $\hat{y}_{t|T} = \boldsymbol{f}' \hat{\boldsymbol{x}}_{t|T}$.
- Useful for seasonal adjustment when one of the states is a seasonal component.

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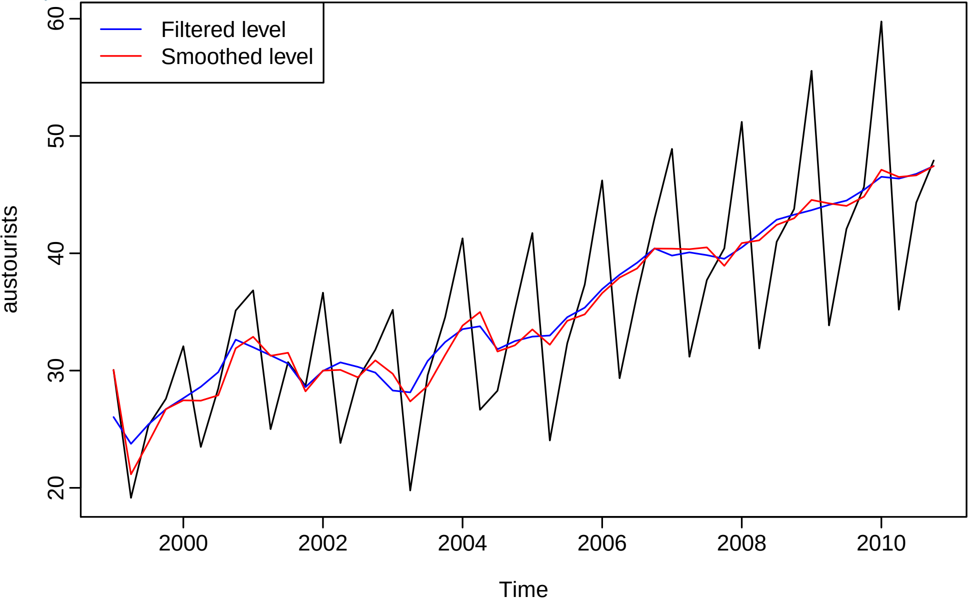
Kalman smoothing in R

```
fit <- StructTS(austourists, type = "BSM")
sm <- tsSmooth(fit)

plot(austourists)
lines(sm[,1],col='blue')
lines(fitted(fit)[,1],col='red')
legend("topleft",col=c('blue','red'),lty=1,
  legend=c("Filtered level","Smoothed level"))
```

Kalman smoothing in R



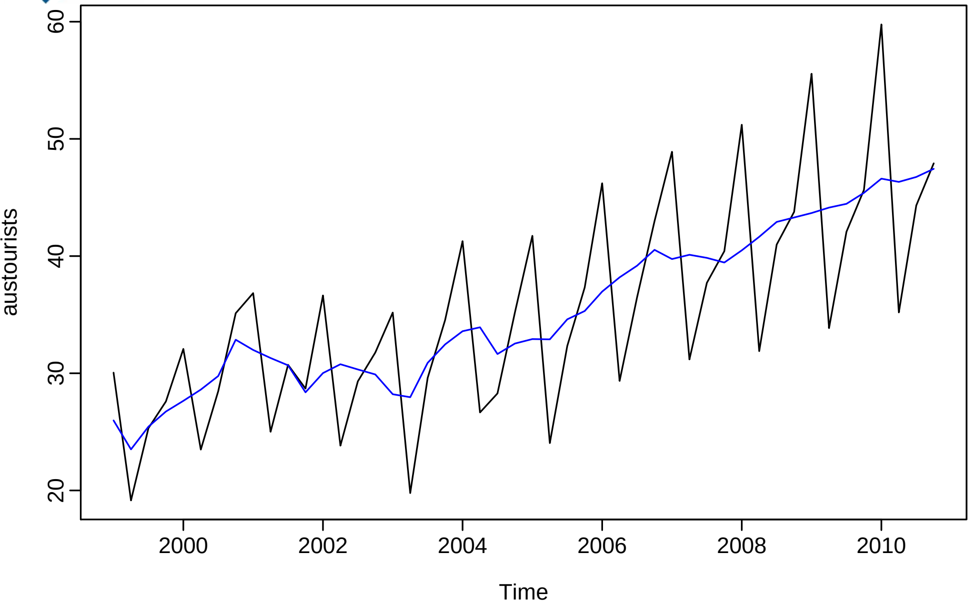
Kalman smoothing in R

```
fit <- StructTS(austourists, type = "BSM")
sm <- tsSmooth(fit)

plot(austourists)

# Seasonally adjusted data
aus.sa <- austourists - sm[,3]
lines(aus.sa,col='blue')
```

Kalman smoothing in R

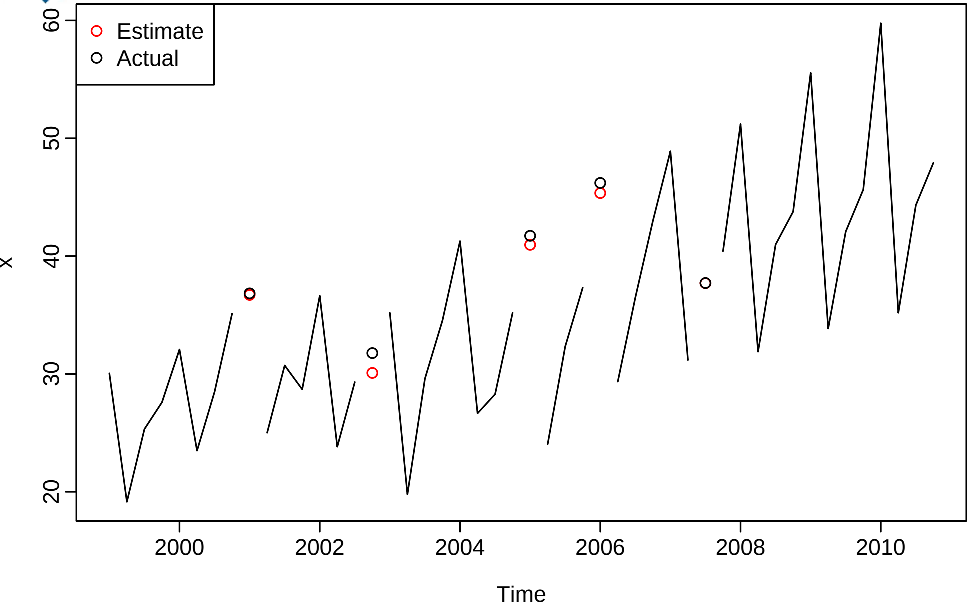


Kalman smoothing in R

```
x <- austourists
miss <- sample(1:length(x), 5)
x[miss] <- NA
fit <- StructTS(x, type = "BSM")
sm <- tsSmooth(fit)
estim <- sm[,1]+sm[,3]

plot(x, ylim=range(austourists))
points(time(x)[miss], estim[miss],
  col='red', pch=1)
points(time(x)[miss], austourists[miss],
  col='black', pch=1)
legend("topleft", pch=1, col=c(2,1),
  legend=c("Estimate","Actual"))
```

Kalman smoothing in R



Outline

1. Simple structural models
2. Linear Gaussian state space models
3. Kalman filter
4. Kalman smoothing
5. **Time varying parameter models**

Time varying parameter models

Linear Gaussian state space model

$$y_t = \boldsymbol{f}_t'\boldsymbol{x}_t + \varepsilon_t, \qquad \varepsilon_t \sim \mathrm{N}(0, \sigma_t^2)$$
$$\boldsymbol{x}_t = \boldsymbol{G}_t\boldsymbol{x}_{t-1} + \boldsymbol{w}_t \qquad \boldsymbol{w}_t \sim \mathrm{N}(\boldsymbol{0}, \boldsymbol{W}_t)$$

Kalman recursions:

$$\hat{y}_{t|t-1} = \boldsymbol{f}_t'\hat{\boldsymbol{x}}_{t|t-1}$$
$$\hat{v}_{t|t-1} = \boldsymbol{f}_t'\hat{\boldsymbol{P}}_{t|t-1}\boldsymbol{f}_t + \sigma_t^2$$
$$\hat{\boldsymbol{x}}_{t|t} = \hat{\boldsymbol{x}}_{t|t-1} + \hat{\boldsymbol{P}}_{t|t-1}\boldsymbol{f}_t\hat{v}_{t|t-1}^{-1}(y_t - \hat{y}_{t|t-1})$$
$$\hat{\boldsymbol{P}}_{t|t} = \hat{\boldsymbol{P}}_{t|t-1} - \hat{\boldsymbol{P}}_{t|t-1}\boldsymbol{f}_t\hat{v}_{t|t-1}^{-1}\boldsymbol{f}_t'\hat{\boldsymbol{P}}_{t|t-1}$$
$$\hat{\boldsymbol{x}}_{t|t-1} = \boldsymbol{G}_t\hat{\boldsymbol{x}}_{t-1|t-1}$$
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Time varying parameter models

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Structural models with covariates

Local level with covariate

$$y_t = \ell_t + \beta z_t + \varepsilon_t$$
$$\ell_t = \ell_{t-1} + \xi_t$$

$$\boldsymbol{f}_t' = [1\ z_t] \quad \boldsymbol{x}_t = \begin{bmatrix} \ell_t \\ \beta \end{bmatrix} \quad \boldsymbol{G} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \boldsymbol{W}_t = \begin{bmatrix} \sigma_\xi^2 & 0 \\ 0 & 0 \end{bmatrix}$$

- Assumes z_t is fixed and known (as in regression)
- Estimate of β is given by $x_{T|T}$.
- Equivalent to simple linear regression with time varying intercept.
- Easy to extend to multiple regression with additional terms.

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Time varying regression

Simple linear regression with time varying parameters

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- Allows for a linear regression with parameters that change slowly over time.
- Parameters follow independent random
- Estimates of parameters given by

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- Allows for a linear regression with parameters that change slowly over time.
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Updating ("online") regression

- Same idea can be used to estimate a regression iteratively as new data arrives.

Simple linear regression with updating parameters

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