State space models

1: Exponential smoothing
1 The state space perspective
2 Simple exponential smoothing
3 Trend methods
4 Seasonal methods
5 Taxonomy of exponential smoothing methods
6 Innovations state space models
7 ETS in R
State space perspective

- **Observed data:** $y_1, \ldots, y_T$.
- **Unobserved state:** $x_1, \ldots, x_T$.

- Forecast $\hat{y}_{T+h|T} = E(y_{T+h}|x_T)$.
- The “forecast variance” is $\text{Var}(y_{T+h}|x_T)$.
- A prediction interval or “interval forecast” is a range of values of $y_{T+h}$ with high probability.
State space perspective

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### Component form

**Forecast equation**  
\[ \hat{y}_{t+h|t} = l_t \]

**Smoothing equation**  
\[ l_t = \alpha y_t + (1 - \alpha) l_{t-1} \]

\[ l_1 = \alpha y_1 + (1 - \alpha) l_0 \]
**Simple Exponential Smoothing**

### Component form

<table>
<thead>
<tr>
<th></th>
<th>Forecast equation</th>
<th>Smoothing equation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \hat{y}_{t+h</td>
<td>t} = \ell_t )</td>
</tr>
</tbody>
</table>

\[ \ell_1 = \alpha y_1 + (1 - \alpha)\ell_0 \]
Simple Exponential Smoothing

Component form

Forecast equation
\[ \hat{y}_{t+h|t} = \ell_t \]

Smoothing equation
\[ \ell_t = \alpha y_t + (1 - \alpha)\ell_{t-1} \]

\[ \ell_1 = \alpha y_1 + (1 - \alpha)\ell_0 \]

\[ \ell_2 = \alpha y_2 + (1 - \alpha)\ell_1 = \alpha y_2 + \alpha(1 - \alpha)y_1 + (1 - \alpha)^2\ell_0 \]
### Component form

**Forecast equation**
\[
\hat{y}_{t+h|t} = \ell_t
\]

**Smoothing equation**
\[
\ell_t = \alpha y_t + (1 - \alpha)\ell_{t-1}
\]

\[
\ell_1 = \alpha y_1 + (1 - \alpha)\ell_0
\]
\[
\ell_2 = \alpha y_2 + (1 - \alpha)\ell_1 = \alpha y_2 + \alpha(1 - \alpha)y_1 + (1 - \alpha)^2\ell_0
\]
\[
\ell_3 = \alpha y_3 + (1 - \alpha)\ell_2 = \sum_{j=0}^{2} \alpha(1 - \alpha)^jy_{3-j} + (1 - \alpha)^3\ell_0
\]
Simple Exponential Smoothing

Component form

Forecast equation: \( \hat{y}_{t+h|t} = \ell_t \)

Smoothing equation: \( \ell_t = \alpha y_t + (1 - \alpha)\ell_{t-1} \)

\[
\begin{align*}
\ell_1 &= \alpha y_1 + (1 - \alpha)\ell_0 \\
\ell_2 &= \alpha y_2 + (1 - \alpha)\ell_1 = \alpha y_2 + \alpha(1 - \alpha)y_1 + (1 - \alpha)^2\ell_0 \\
\ell_3 &= \alpha y_3 + (1 - \alpha)\ell_2 = \sum_{j=0}^{2} \alpha(1 - \alpha)^j y_{3-j} + (1 - \alpha)^3\ell_0 \\
&\vdots \\
\ell_t &= \sum_{j=0}^{t-1} \alpha(1 - \alpha)^j y_{t-j} + (1 - \alpha)^t\ell_0
\end{align*}
\]
Simple Exponential Smoothing

Forecast equation

\[ \hat{y}_{t+h|t} = \sum_{j=1}^{t} \alpha(1 - \alpha)^{t-j}y_j + (1 - \alpha)^t \ell_0, \quad (0 \leq \alpha \leq 1) \]

<table>
<thead>
<tr>
<th>Observation</th>
<th>Weights assigned to observations for:</th>
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<tbody>
<tr>
<td></td>
<td>( \alpha = 0.2 )</td>
</tr>
<tr>
<td>( y_t )</td>
<td>0.2</td>
</tr>
<tr>
<td>( y_{t-1} )</td>
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</tr>
<tr>
<td>( y_{t-2} )</td>
<td>0.128</td>
</tr>
<tr>
<td>( y_{t-3} )</td>
<td>0.1024</td>
</tr>
<tr>
<td>( y_{t-4} )</td>
<td>((0.2)(0.8)^4)</td>
</tr>
<tr>
<td>( y_{t-5} )</td>
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Limiting cases: \( \alpha \to 1 \), \( \alpha \to 0 \).
**Simple Exponential Smoothing**

**Forecast equation**

\[
\hat{y}_{t+h|t} = \sum_{j=1}^{t} \alpha (1 - \alpha)^{t-j} y_j + (1 - \alpha)^t \ell_0, \quad (0 \leq \alpha \leq 1)
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<tr>
<td>( y_t )</td>
<td>0.2</td>
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</tr>
<tr>
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**Limiting cases:** \( \alpha \to 1, \quad \alpha \to 0 \).
Simple Exponential Smoothing

Forecast equation

\[ \hat{y}_{t+h|t} = \sum_{j=1}^{t} \alpha (1 - \alpha)^{t-j} y_j + (1 - \alpha)^t \ell_0, \quad (0 \leq \alpha \leq 1) \]

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- Limiting cases: \( \alpha \to 1, \quad \alpha \to 0. \)
Simple Exponential Smoothing

Component form

Forecast equation
\[ \hat{y}_{t+h|t} = \ell_t \]

Smoothing equation
\[ \ell_t = \alpha y_t + (1 - \alpha) \ell_{t-1} \]

State space form

Observation equation
\[ y_t = \ell_{t-1} + e_t \]

State equation
\[ \ell_t = \ell_{t-1} + \alpha e_t \]

- \( e_t = y_t - \ell_{t-1} = y_t - \hat{y}_{t|t-1} \) for \( t = 1, \ldots, T \), the one-step within-sample forecast error at time \( t \).
- \( \ell_t \) is an unobserved “state”.
- Need to estimate \( \alpha \) and \( \ell_0 \).
**Simple Exponential Smoothing**

### Component form

| Forecast equation  | $\hat{y}_{t+h|t} = \ell_t$ |
|---------------------|-----------------------------|
| Smoothing equation  | $\ell_t = \alpha y_t + (1 - \alpha)\ell_{t-1}$ |

### State space form

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<tr>
<th>Observation equation</th>
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<tbody>
<tr>
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- $e_t = y_t - \ell_{t-1} = y_t - \hat{y}_{t|t-1}$ for $t = 1, \ldots, T$, the one-step within-sample forecast error at time $t$.
- $\ell_t$ is an unobserved “state”.
- Need to estimate $\alpha$ and $\ell_0$. 

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**State space models**

| 1: Exponential smoothing | 8 |
# Simple Exponential Smoothing

## Component form

### Forecast equation

\[ \hat{y}_{t+h|t} = l_t \]

### Smoothing equation

\[ l_t = \alpha y_t + (1 - \alpha)l_{t-1} \]

## State space form

### Observation equation

\[ y_t = l_{t-1} + e_t \]

### State equation

\[ l_t = l_{t-1} + \alpha e_t \]

- \( e_t = y_t - l_{t-1} = y_t - \hat{y}_{t|t-1} \) for \( t = 1, \ldots, T \), the one-step within-sample forecast error at time \( t \).
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## Simple Exponential Smoothing

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- \( \ell_t \) is an unobserved “state”.
- Need to estimate \( \alpha \) and \( \ell_0 \).
library(fpp)

fit <- ses(oil, h=3)

plot(fit)

summary(fit)
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Holt’s linear trend

Component form

Forecast \( \hat{y}_{t+h|t} = \ell_t + h b_t \)

Level \( \ell_t = \alpha y_t + (1 - \alpha)(\ell_{t-1} + b_{t-1}) \)

Trend \( b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1}, \)

- Two smoothing parameters \( \alpha \) and \( \beta^* \) (\( 0 \leq \alpha, \beta^* \leq 1 \)).
- \( \ell_t \) level: weighted average between \( y_t \) one-step ahead forecast for time \( t \), \( (\ell_{t-1} + b_{t-1} = \hat{y}_{t|t-1}) \)
- \( b_t \) trend (slope): weighted average of \( (\ell_t - \ell_{t-1}) \) and \( b_{t-1} \), current and previous estimate of the trend.
### Holt’s linear trend

#### Component form

<table>
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<tr>
<th>Component</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forecast</td>
<td>$\hat{y}_{t+h</td>
</tr>
<tr>
<td>Level</td>
<td>$\ell_t = \alpha y_t + (1 - \alpha)(\ell_{t-1} + b_{t-1})$</td>
</tr>
<tr>
<td>Trend</td>
<td>$b_t = \beta^<em>(\ell_t - \ell_{t-1}) + (1 - \beta^</em>)b_{t-1}$,</td>
</tr>
</tbody>
</table>

- **Two smoothing parameters** $\alpha$ and $\beta^*$ ($0 \leq \alpha, \beta^* \leq 1$).

- $\ell_t$ level: weighted average between $y_t$ one-step ahead forecast for time $t$, ($\ell_{t-1} + b_{t-1} = \hat{y}_{t|t-1}$)

- $b_t$ trend (slope): weighted average of $(\ell_t - \ell_{t-1})$ and $b_{t-1}$, current and previous estimate of the trend.
Holt’s linear trend

Component form

Forecast \[ \hat{y}_{t+h|t} = \ell_t + hb_t \]

Level \[ \ell_t = \alpha y_t + (1 - \alpha)(\ell_{t-1} + b_{t-1}) \]

Trend \[ b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1}, \]

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### Holt’s linear trend

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| Forecast  | $\hat{y}_{t+h|t} = \ell_t + hb_t$ |
|-----------|----------------------------------|
| Level     | $\ell_t = \alpha y_t + (1 - \alpha)(\ell_{t-1} + b_{t-1})$ |
| Trend     | $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1},$ |

- Two smoothing parameters $\alpha$ and $\beta^*$
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Holt’s linear trend

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Forecast  \[ \hat{y}_{t+h|t} = \ell_t + hb_t \]

Level  \[ \ell_t = \alpha y_t + (1 - \alpha)(\ell_{t-1} + b_{t-1}) \]

Trend  \[ b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1}, \]

State space form

Observation equation  \[ y_t = \ell_{t-1} + b_{t-1} + e_t \]

State equations  \[ \ell_t = \ell_{t-1} + b_{t-1} + \alpha e_t \]
\[ b_t = b_{t-1} + \beta e_t \]

- \( \beta = \alpha \beta^* \)
- \( e_t = y_t - (\ell_{t-1} + b_{t-1}) = y_t - \hat{y}_{t|t-1} \)
- Need to estimate \( \alpha, \beta, \ell_0, b_0 \)
Holt’s linear trend

**Component form**

Forecast\(\hat{y}_{t+h|t} = \ell_t + hb_t\)

Level\(\ell_t = \alpha y_t + (1 - \alpha)(\ell_{t-1} + b_{t-1})\)

Trend\(b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1},\)

**State space form**

Observation equation\(y_t = \ell_{t-1} + b_{t-1} + e_t\)

State equations\(\ell_t = \ell_{t-1} + b_{t-1} + \alpha e_t\)
\(b_t = b_{t-1} + \beta e_t\)

- \(\beta = \alpha \beta^*\)
- \(e_t = y_t - (\ell_{t-1} + b_{t-1}) = y_t - \hat{y}_{t|t-1}\)
- Need to estimate \(\alpha, \beta, \ell_0, b_0.\)
Holt’s linear trend

Component form

Forecast \[ \hat{y}_{t+h|t} = \ell_t + hb_t \]
Level \[ \ell_t = \alpha y_t + (1 - \alpha)(\ell_{t-1} + b_{t-1}) \]
Trend \[ b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1}, \]

State space form

Observation equation \[ y_t = \ell_{t-1} + b_{t-1} + e_t \]
State equations \[ \ell_t = \ell_{t-1} + b_{t-1} + \alpha e_t \]
\[ b_t = b_{t-1} + \beta e_t \]

\[ \beta = \alpha \beta^* \]
\[ e_t = y_t - (\ell_{t-1} + b_{t-1}) = y_t - \hat{y}_{t|t-1} \]
Need to estimate \( \alpha, \beta, \ell_0, b_0 \).
Holt’s linear trend

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- $\beta = \alpha \beta^*$
- $e_t = y_t - (\ell_{t-1} + b_{t-1}) = y_t - \hat{y}_{t|t-1}$
- Need to estimate $\alpha, \beta, \ell_0, b_0$. 

State space models
Holt’s method in R

```r
fit2 <- holt(ausair, h=5)
plot(fit2)
summary(fit2)
```
Exponential trend

Level and trend are multiplied rather than added:

**Component form**

\[
\hat{y}_{t+h|t} = \ell_t b_t^h \\
\ell_t = \alpha y_t + (1 - \alpha)(\ell_{t-1} b_{t-1}) \\
b_t = \beta^* \frac{\ell_t}{\ell_{t-1}} + (1 - \beta^*)b_{t-1}
\]

**State space form**

Observation equation

\[ y_t = (\ell_{t-1} b_{t-1}) + e_t \]

State equations

\[ \ell_t = \ell_{t-1} b_{t-1} + \alpha e_t \]
\[ b_t = b_{t-1} + \beta e_t / \ell_{t-1} \]
Trend methods in R

fit3 <- holt(air, h=5, exponential=TRUE)

plot(fit3)

summary(fit3)
Additive damped trend

Component form

\[
\hat{y}_{t+h | t} = \ell_t + (\phi + \phi^2 + \cdots + \phi^h)b_t
\]
\[
\ell_t = \alpha y_t + (1 - \alpha) (\ell_{t-1} + \phi b_{t-1})
\]
\[
b_t = \beta^* (\ell_t - \ell_{t-1}) + (1 - \beta^*) \phi b_{t-1}.
\]

State space form

Observation equation

\[
y_t = \ell_{t-1} + \phi b_{t-1} + e_t
\]

State equations

\[
\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha e_t
\]
\[
b_t = \phi b_{t-1} + \beta e_t
\]

- Damping parameter 0 < \(\phi\) < 1.
- If \(\phi = 1\), identical to Holt’s linear trend.
- As \(h \to \infty\), \(\hat{y}_{T+h | t} \to \ell_T + \phi b_T /(1 - \phi)\).
- Short-run forecasts trended, long-run forecasts constant.

State space models

1: Exponential smoothing
Additive damped trend

**Component form**

\[
\hat{y}_{t+h|t} = \ell_t + (\phi + \phi^2 + \cdots + \phi^h)b_t
\]

\[
\ell_t = \alpha y_t + (1 - \alpha)(\ell_{t-1} + \phi b_{t-1})
\]

\[
b_t = \beta^* (\ell_t - \ell_{t-1}) + (1 - \beta^*)\phi b_{t-1}.
\]

**State space form**

**Observation equation**

\[
y_t = \ell_{t-1} + \phi b_{t-1} + e_t
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**State equations**

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\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha e_t
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Additive damped trend

**Component form**

\[ \hat{y}_{t+h|t} = \ell_t + (\phi + \phi^2 + \cdots + \phi^h)b_t \]
\[ \ell_t = \alpha y_t + (1 - \alpha)(\ell_{t-1} + \phi b_{t-1}) \]
\[ b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)\phi b_{t-1}. \]

**State space form**

**Observation equation**

\[ y_t = \ell_{t-1} + \phi b_{t-1} + e_t \]

**State equations**

\[ \ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha e_t \]
\[ b_t = \phi b_{t-1} + \beta e_t \]

- Damping parameter \(0 < \phi < 1\).
- If \(\phi = 1\), identical to Holt’s linear trend.
- As \(h \to \infty\), \(\hat{y}_{T+h|T} \to \ell_T + \phi b_T/(1 - \phi)\).
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Additive damped trend

**Component form**

\[ \hat{y}_{t+h|t} = \ell_t + (\phi + \phi^2 + \cdots + \phi^h)b_t \]
\[ \ell_t = \alpha y_t + (1 - \alpha)(\ell_{t-1} + \phi b_{t-1}) \]
\[ b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)\phi b_{t-1}. \]

**State space form**

Observation equation

\[ y_t = \ell_{t-1} + \phi b_{t-1} + e_t \]

State equations

\[ \ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha e_t \]
\[ b_t = \phi b_{t-1} + \beta e_t \]

- Damping parameter \(0 < \phi < 1\).
- If \(\phi = 1\), identical to Holt’s linear trend.
- As \(h \to \infty\), \(\hat{y}_{T+h|T} \to \ell_T + \phi b_T/(1 - \phi)\).

Short-run forecasts trended, long-run forecasts constant.
Additive damped trend

Component form

\[ \hat{y}_{t+h|t} = \ell_t + (\phi + \phi^2 + \cdots + \phi^h)b_t \]
\[ \ell_t = \alpha y_t + (1 - \alpha)(\ell_{t-1} + \phi b_{t-1}) \]
\[ b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)\phi b_{t-1}. \]

State space form

Observation equation
\[ y_t = \ell_{t-1} + \phi b_{t-1} + e_t \]
State equations
\[ \ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha e_t \]
\[ b_t = \phi b_{t-1} + \beta e_t \]

- Damping parameter \( 0 < \phi < 1. \)
- If \( \phi = 1, \) identical to Holt’s linear trend.
- As \( h \to \infty, \) \( \hat{y}_{T+h|T} \to \ell_T + \phi b_T/(1 - \phi). \)
- Short-run forecasts trended, long-run forecasts constant.
fit4 <- holt(air, h=5, damped=TRUE)
plot(fit4)
summary(fit4)
Example: Sheep in Asia

Forecasts from Holt's method with exponential trend

- Data
- SES
- Holt's
- Exponential
- Additive Damped
- Multiplicative Damped

Livestock, sheep in Asia (millions)


300 350 400 450

State space models
1: Exponential smoothing
1 The state space perspective
2 Simple exponential smoothing
3 Trend methods
4 Seasonal methods
5 Taxonomy of exponential smoothing methods
6 Innovations state space models
7 ETS in R
Holt-Winters additive method

Holt and Winters extended Holt’s method to capture seasonality.

Component form

\[
\hat{y}_{t+h|t} = \ell_t + hb_t + s_{t-m+h^+_m}
\]

\[
\ell_t = \alpha(y_t - s_{t-m}) + (1 - \alpha)(\ell_{t-1} + b_{t-1})
\]

\[
b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1}
\]

\[
s_t = \gamma(y_t - \ell_{t-1} - b_{t-1}) + (1 - \gamma)s_{t-m},
\]

\[
h^+_m = [(h - 1) \mod m] + 1 - \text{the largest integer not greater than } (h - 1) \mod m. \text{ Ensures estimates from the final year are used for forecasting.}
\]

Parameters: 0 ≤ α ≤ 1, 0 ≤ β* ≤ 1, 0 ≤ γ ≤ 1 − α and m = period of seasonality (e.g. m=4 for quarterly data).
Holt-Winters additive method

Holt and Winters extended Holt’s method to capture seasonality.

Component form

\[ \hat{y}_{t+h|t} = l_t + hb_t + s_{t-m+h_m^+} \]
\[ l_t = \alpha(y_t - s_{t-m}) + (1 - \alpha)(l_{t-1} + b_{t-1}) \]
\[ b_t = \beta^*(l_t - l_{t-1}) + (1 - \beta^*)b_{t-1} \]
\[ s_t = \gamma(y_t - l_{t-1} - b_{t-1}) + (1 - \gamma)s_{t-m}, \]

- \( h_m^+ = \lfloor (h - 1) \mod m \rfloor + 1 \) - the largest integer not greater than \((h - 1) \mod m\). Ensures estimates from the final year are used for forecasting.

- Parameters: \( 0 \leq \alpha \leq 1, \ 0 \leq \beta^* \leq 1, \ 0 \leq \gamma \leq 1 - \alpha \) and \( m = \) period of seasonality (e.g. \( m=4 \) for quarterly data).
Holt-Winters additive method

Holt and Winters extended Holt’s method to capture seasonality.

Component form

\[
\hat{y}_{t+h|t} = \ell_t + hb_t + s_{t-m+h_m^+}
\]

\[
\ell_t = \alpha(y_t - s_{t-m}) + (1 - \alpha)(\ell_{t-1} + b_{t-1})
\]

\[
b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1}
\]

\[
s_t = \gamma(y_t - \ell_{t-1} - b_{t-1}) + (1 - \gamma)s_{t-m},
\]

- \(h_m^+ = \lfloor (h - 1) \mod m \rfloor + 1\) - the largest integer not greater than \((h - 1) \mod m\). Ensures estimates from the final year are used for forecasting.

- Parameters: \(0 \leq \alpha \leq 1\), \(0 \leq \beta^* \leq 1\), \(0 \leq \gamma \leq 1 - \alpha\) and \(m = \) period of seasonality (e.g. \(m=4\) for quarterly data).
Holt-Winters additive method

**Component form**

\[ \hat{y}_{t+h|t} = \ell_t + h b_t + s_{t-m+h_m} \]
\[ \ell_t = \alpha(y_t - s_{t-m}) + (1 - \alpha)(\ell_{t-1} + b_{t-1}) \]
\[ b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1} \]
\[ s_t = \gamma(y_t - \ell_{t-1} - b_{t-1}) + (1 - \gamma)s_{t-m}, \]

**State space form**

\[ y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + e_t \]
\[ \ell_t = \ell_{t-1} + b_{t-1} + \alpha e_t \]
\[ b_t = b_{t-1} + \beta e_t \]
\[ s_t = s_{t-m} + \gamma e_t. \]
Holt-Winters multiplicative

**Component form**

\[
\hat{y}_{t+h|t} = (\ell_t + hb_t)s_{t-m+h^+_m}.
\]

\[
\ell_t = \alpha \frac{y_t}{s_{t-m}} + (1 - \alpha)(\ell_{t-1} + b_{t-1})
\]

\[
b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1}
\]

\[
s_t = \gamma \frac{y_t}{(\ell_{t-1} + b_{t-1})} + (1 - \gamma)s_{t-m}
\]

**State space form**

\[
y_t = (\ell_{t-1} + b_{t-1})s_{t-m} + e_t
\]

\[
\ell_t = \ell_{t-1} + b_{t-1} + \alpha e_t/s_{t-m}
\]

\[
b_t = b_{t-1} + \beta e_t/s_{t-m}
\]

\[
s_t = s_{t-m} + \gamma e_t/(\ell_{t-1} + b_{t-1}).
\]
Seasonal methods in R

```r
aus1 <- hw(austourists)
aus2 <- hw(austourists, seasonal="mult")

plot(aus1)
plot(aus2)

summary(aus1)
summary(aus2)
```
Holt-Winters damped method

Often the single most accurate forecasting method for seasonal data:

\[
\hat{y}_{t+h|t} = [\ell_t + (\phi + \phi^2 + \cdots + \phi^h) b_t] s_{t-m+h^+} \\
\ell_t = \alpha (y_t / s_{t-m}) + (1 - \alpha)(\ell_{t-1} + \phi b_{t-1}) \\
b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*) \phi b_{t-1} \\
s_t = \gamma \frac{y_t}{(\ell_{t-1} + \phi b_{t-1})} + (1 - \gamma) s_{t-m}
\]

State space form

\[
y_t = (\ell_{t-1} + \phi b_{t-1}) s_{t-m} + e_t \\
\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha e_t / s_{t-m} \\
b_t = \phi b_{t-1} + \beta e_t / s_{t-m} \\
s_t = s_{t-m} + \gamma e_t / (\ell_{t-1} + \phi b_{t-1}).
\]
Seasonal methods in R

```r
aus3 <- hw(austourists, seasonal="mult", damped=TRUE)

summary(aus3)

plot(aus3)
```
Outline

1 The state space perspective
2 Simple exponential smoothing
3 Trend methods
4 Seasonal methods
5 **Taxonomy of exponential smoothing methods**
6 Innovations state space models
7 ETS in R
## Exponential smoothing methods

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### State space models

1: Exponential smoothing
### Exponential smoothing methods

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**N,N:** Simple exponential smoothing
### Exponential smoothing methods

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- **N,N**: Simple exponential smoothing
- **A,N**: Holt’s linear method
### Exponential smoothing methods

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N,N: Simple exponential smoothing  
A,N: Holt’s linear method  
A\textsubscript{d},N: Additive damped trend method
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- **M,N**: Exponential trend method
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There are 15 separate exponential smoothing methods.
## Component form

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<tr>
<td>( \hat{y}_{t+h</td>
<td>t} = \ell_t )</td>
<td>( \hat{y}_{t+h</td>
<td>t} = \ell_t + s_{t-m+h^+_m} )</td>
<td>( \hat{y}_{t+h</td>
</tr>
<tr>
<td>( \ell_t = \alpha y_t + (1 - \alpha) \ell_{t-1} )</td>
<td>( \ell_t = \alpha (y_t - s_{t-m}) + (1 - \alpha) \ell_{t-1} )</td>
<td>( \ell_t = \alpha (y_t/s_{t-m}) + (1 - \alpha) \ell_{t-1} )</td>
<td></td>
<td></td>
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<td>( s_t = \gamma (y_t - \ell_{t-1}) + (1 - \gamma) s_{t-m} )</td>
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### Trend

- **N**
  - \( \hat{y}_{t+h|t} = \ell_t \)
  - \( \ell_t = \alpha y_t + (1 - \alpha) \ell_{t-1} \)
  - \( s_t = \gamma (y_t - \ell_{t-1}) + (1 - \gamma) s_{t-m} \)

### Seasonal

- **A**
  - \( \hat{y}_{t+h|t} = \ell_t + h b_t \)
  - \( \ell_t = \alpha (y_t - s_{t-m}) + (1 - \alpha) (\ell_{t-1} + b_{t-1}) \)
  - \( b_t = \beta^* (\ell_t - \ell_{t-1}) + (1 - \beta^*) b_{t-1} \)
  - \( s_t = \gamma (y_t - \ell_{t-1} - b_{t-1}) + (1 - \gamma) s_{t-m} \)

### Trend (Ad)

- \( \hat{y}_{t+h|t} = \ell_t + \phi_h b_t \)
  - \( \ell_t = \alpha (y_t - s_{t-m}) + (1 - \alpha) (\ell_{t-1} + \phi b_{t-1}) \)
  - \( b_t = \beta^* (\ell_t - \ell_{t-1}) + (1 - \beta^*) \phi b_{t-1} \)
  - \( s_t = \gamma (y_t - \ell_{t-1} - \phi b_{t-1}) + (1 - \gamma) s_{t-m} \)

### Trend (M)

- \( \hat{y}_{t+h|t} = \ell_t b^h_t \)
  - \( \ell_t = \alpha (y_t - s_{t-m}) + (1 - \alpha) \ell_{t-1} b_{t-1} \)
  - \( b_t = \beta^* (\ell_t/\ell_{t-1}) + (1 - \beta^*) b_{t-1} \)
  - \( s_t = \gamma (y_t - \ell_{t-1} b_{t-1}) + (1 - \gamma) s_{t-m} \)

### Trend (Md)

- \( \hat{y}_{t+h|t} = \ell_t b^{\phi h}_t \)
  - \( \ell_t = \alpha (y_t - s_{t-m}) + (1 - \alpha) \ell_{t-1} b^{\phi}_{t-1} \)
  - \( b_t = \beta^* (\ell_t/\ell_{t-1}) + (1 - \beta^*) b^{\phi}_{t-1} \)
  - \( s_t = \gamma (y_t - \ell_{t-1} b^{\phi}_{t-1}) + (1 - \gamma) s_{t-m} \)
Outline

1. The state space perspective
2. Simple exponential smoothing
3. Trend methods
4. Seasonal methods
5. Taxonomy of exponential smoothing methods
6. Innovations state space models
7. ETS in R
Exponential smoothing methods

- Algorithms that return point forecasts.

Innovations state space models

- Generate same point forecasts but can also generate forecast intervals.
- A stochastic (or random) data generating process that can generate an entire forecast distribution.
- Allow for “proper” model selection.
### Methods V Models

#### Exponential smoothing methods
- Algorithms that return point forecasts.

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- Generate same point forecasts but can also generate forecast intervals.
- A stochastic (or random) data generating process that can generate an entire forecast distribution.
- Allow for “proper” model selection.
ETS models

- Each model has an *observation* equation and *transition* equations, one for each state (level, trend, seasonal), i.e., state space models.

- Two models for each method: one with additive and one with multiplicative errors, i.e., in total 30 models.

- ETS(Error,Trend,Seasonal):
  - Error = \{A, M\}
  - Trend = \{N, A, A, M, Md\}
  - Seasonal = \{N, A, M\}.
ETS models

- Each model has an observation equation and transition equations, one for each state (level, trend, seasonal), i.e., state space models.

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### Exponential smoothing methods

<table>
<thead>
<tr>
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<th>Seasonal Component</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>N</td>
</tr>
<tr>
<td></td>
<td>(None)</td>
</tr>
<tr>
<td>N (None)</td>
<td>N,N</td>
</tr>
<tr>
<td>A (Additive)</td>
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**General notation**

E T S : Exponential Smoothing

**Examples:**
- A,N,N: Simple exponential smoothing with additive errors
- A,A,N: Holt's linear method with additive errors
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## Exponential smoothing methods

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### General notation

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### General notation

ETS : **Exponential Smoothing**

↑ ↖

**Trend** **Seasonal**

### Examples:

- **A,N,N**: Simple exponential smoothing with additive errors
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# Exponential smoothing methods

## General notation

ETSM : **Exponential Smoothing**

- **Error**
- **Trend**
- **Seasonal**

## Examples:

A,N,N: Simple exponential smoothing with additive errors
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**General notation**  
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Exponential smoothing methods

Innovations state space models

- All ETS models can be written in innovations state space form.
- Additive and multiplicative versions give the same point forecasts but different prediction intervals.

General notation

ETS : Exponential Smoothing

Error Trend Seasonal

Examples:

A,N,N: Simple exponential smoothing with additive errors
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**ETS(A,N,N)**

**Observation equation**

\[ y_t = l_{t-1} + \varepsilon_t, \]

**State equation**

\[ l_t = l_{t-1} + \alpha \varepsilon_t \]

- \[ e_t = y_t - \hat{y}_{t|t-1} = \varepsilon_t \]
- Assume \( \varepsilon_t \sim NID(0, \sigma^2) \)
- “innovations” or “single source of error” because same error process, \( \varepsilon_t \).
ETS(A,N,N)

Observation equation  \( y_t = \ell_{t-1} + \varepsilon_t, \)

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ETS(M,N,N)

SES with multiplicative errors.

- Specify relative errors $\varepsilon_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}} \sim \text{NID}(0, \sigma^2)$

- Substituting $\hat{y}_{t|t-1} = l_{t-1}$ gives:
  - $y_t = l_{t-1} + l_{t-1}\varepsilon_t$
  - $e_t = y_t - \hat{y}_{t|t-1} = l_{t-1}\varepsilon_t$

**Observation equation**

$y_t = l_{t-1}(1 + \varepsilon_t)$

**State equation**

$l_t = l_{t-1}(1 + \alpha\varepsilon_t)$
ETS(M,N,N)

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Observation equation $y_t = l_{t-1}(1 + \varepsilon_t)$
State equation $l_t = l_{t-1}(1 + \alpha \varepsilon_t)$

Models with additive and multiplicative errors with the same parameters generate the same point forecasts but different prediction intervals.
ETS(M,N,N)

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Models with additive and multiplicative errors with the same parameters generate the same point forecasts but different prediction intervals.
ETS(M,N,N)

SERIES with multiplicative errors.

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Observation equation: $y_t = l_{t-1}(1 + \varepsilon_t)$
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State space models
ETS(M,N,N)

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- Models with additive and multiplicative errors with the same parameters generate the same point forecasts but different prediction intervals.
Holt’s linear method

ETS(A,A,N)

\[ y_t = \ell_{t-1} + b_{t-1} + \varepsilon_t \]
\[ \ell_t = \ell_{t-1} + b_{t-1} + \alpha \varepsilon_t \]
\[ b_t = b_{t-1} + \beta \varepsilon_t \]

ETS(M,A,N)

\[ y_t = (\ell_{t-1} + b_{t-1})(1 + \varepsilon_t) \]
\[ \ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha \varepsilon_t) \]
\[ b_t = b_{t-1} + \beta (\ell_{t-1} + b_{t-1}) \varepsilon_t \]
Holt’s linear method

**ETS(A,A,N)**

\[
y_t = l_{t-1} + b_{t-1} + \varepsilon_t \\
l_t = l_{t-1} + b_{t-1} + \alpha \varepsilon_t \\
b_t = b_{t-1} + \beta \varepsilon_t
\]

**ETS(M,A,N)**

\[
y_t = (l_{t-1} + b_{t-1})(1 + \varepsilon_t) \\
l_t = (l_{t-1} + b_{t-1})(1 + \alpha \varepsilon_t) \\
b_t = b_{t-1} + \beta (l_{t-1} + b_{t-1}) \varepsilon_t
\]
ETS(A,A,A)

Holt-Winters additive method with additive errors.

Forecast equation
\[ \hat{y}_{t+h|t} = \ell_t + h b_t + s_{t-m+h^+_m} \]

Observation equation
\[ y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \epsilon_t \]

State equations
\[ \ell_t = \ell_{t-1} + b_{t-1} + \alpha \epsilon_t \]
\[ b_t = b_{t-1} + \beta \epsilon_t \]
\[ s_t = s_{t-m} + \gamma \epsilon_t \]

- Forecast errors: \( \epsilon_t = y_t - \hat{y}_{t|t-1} \)
- \( h^+_m = \left\lfloor (h - 1) \mod m \right\rfloor + 1 \).
### Additive error models

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<td><strong>A</strong></td>
<td>$y_t = \ell_{t-1} + b_{t-1} + \varepsilon_t$</td>
<td>$y_t = (\ell_{t-1} + b_{t-1}) s_{t-m} + \varepsilon_t$</td>
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<td>$b_t = b_{t-1} + \beta \varepsilon_t/\ell_{t-1}$</td>
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<td><strong>M_d</strong></td>
<td>$y_t = \ell_{t-1} b_{t-1}^\phi + \varepsilon_t$</td>
<td>$y_t = \ell_{t-1} b_{t-1}^\phi s_{t-m} + \varepsilon_t$</td>
</tr>
<tr>
<td></td>
<td>$\ell_t = \ell_{t-1} b_{t-1}^\phi + \alpha \varepsilon_t$</td>
<td>$\ell_t = \ell_{t-1} b_{t-1}^\phi + \alpha \varepsilon_t/s_{t-m}$</td>
</tr>
<tr>
<td></td>
<td>$b_t = b_{t-1}^\phi + \beta \varepsilon_t/\ell_{t-1}$</td>
<td>$b_t = b_{t-1}^\phi + \beta \varepsilon_t/(s_{t-m}\ell_{t-1})$</td>
</tr>
</tbody>
</table>
## Additive Error Models

For each of the models in the ETS framework:

### Trend

<table>
<thead>
<tr>
<th>N</th>
<th>Seasonal</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y_t = \ell_{t-1}(1 + \varepsilon_t))</td>
<td>(y_t = (\ell_{t-1} + s_{t-m})(1 + \varepsilon_t))</td>
<td>(y_t = \ell_{t-1}s_{t-m}(1 + \varepsilon_t))</td>
</tr>
<tr>
<td>(\ell_t = \ell_{t-1}(1 + \alpha \varepsilon_t))</td>
<td>(\ell_t = \ell_{t-1} + \alpha(\ell_{t-1} + s_{t-m})\varepsilon_t)</td>
<td>(\ell_t = \ell_{t-1}(1 + \alpha \varepsilon_t))</td>
</tr>
<tr>
<td>(s_t = s_{t-m} + \gamma(\ell_{t-1} + s_{t-m})\varepsilon_t)</td>
<td>(s_t = s_{t-m}(1 + \gamma \varepsilon_t))</td>
<td></td>
</tr>
</tbody>
</table>

### Seasonal

<table>
<thead>
<tr>
<th>A</th>
<th>A_d</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y_t = (\ell_{t-1} + b_{t-1})(1 + \varepsilon_t))</td>
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<td>(s_t = s_{t-m}(1 + \gamma \varepsilon_t))</td>
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### Multiplicative Error Models

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<th>M_d</th>
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<tr>
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<td>(y_t = (\ell_{t-1}b_{t-1} + s_{t-m})(1 + \varepsilon_t))</td>
</tr>
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<td>(\ell_t = \ell_{t-1}b_{t-1}(1 + \alpha \varepsilon_t))</td>
<td>(\ell_t = \ell_{t-1}b_{t-1} + \alpha(\ell_{t-1}b_{t-1} + s_{t-m})\varepsilon_t)</td>
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<td>(b_t = b_{t-1}(1 + \beta \varepsilon_t))</td>
<td>(b_t = b_{t-1} + \beta(\ell_{t-1}b_{t-1} + s_{t-m})\varepsilon_t/\ell_{t-1})</td>
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---

### State Space Models

1: Exponential smoothing
Innovations state space models

Let $\mathbf{x}_t = (\ell_t, b_t, s_t, s_{t-1}, \ldots, s_{t-m+1})$ and $\varepsilon_t \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$.

$$
\begin{align*}
\mathbf{y}_t &= h(\mathbf{x}_{t-1}) + k(\mathbf{x}_{t-1})\varepsilon_t \\
\mu_t &= \mu_t \\
\mathbf{e}_t &= e_t \\
\mathbf{x}_t &= f(\mathbf{x}_{t-1}) + g(\mathbf{x}_{t-1})\varepsilon_t
\end{align*}
$$

Additive errors:

$$k(x) = 1. \quad y_t = \mu_t + \varepsilon_t.$$

Multiplicative errors:

$$k(\mathbf{x}_{t-1}) = \mu_t. \quad y_t = \mu_t(1 + \varepsilon_t).$$

$$\varepsilon_t = \frac{(y_t - \mu_t)}{\mu_t} \text{ is relative error.}$$
All the methods can be written in this state space form.

The only difference between the additive error and multiplicative error models is in the observation equation.

Additive and multiplicative versions give the same point forecasts.
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Some unstable models

Some of the combinations of (Error, Trend, Seasonal) can lead to numerical difficulties; see equations with division by a state.

These are: $\text{ETS}(M,M,A)$, $\text{ETS}(M,M_d,A)$, $\text{ETS}(A,N,M)$, $\text{ETS}(A,A,M)$, $\text{ETS}(A,A_d,M)$, $\text{ETS}(A,M,N)$, $\text{ETS}(A,M,A)$, $\text{ETS}(A,M,M)$, $\text{ETS}(A,M_d,N)$, $\text{ETS}(A,M_d,A)$, and $\text{ETS}(A,M_d,M)$.

Models with multiplicative errors are useful for strictly positive data – but are not numerically stable with data containing zeros or negative values. In that case only the six fully additive models will be applied.
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</tr>
<tr>
<td>Trend Component</td>
<td></td>
</tr>
<tr>
<td>N (None)</td>
<td>A,N,N,N</td>
</tr>
<tr>
<td>A (Additive)</td>
<td>A,A,N</td>
</tr>
<tr>
<td>(A_d) (Additive damped)</td>
<td>A,A_d,N</td>
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<td>(\Lambda,M,N)</td>
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Innovations state space models

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\[ L^*(\theta, x_0) = n \log\left( \sum_{t=1}^{n} \frac{\varepsilon_t^2}{k^2(x_{t-1})} \right) + 2 \sum_{t=1}^{n} \log |k(x_{t-1})| \]

\[ = -2 \log(\text{Likelihood}) + \text{constant} \]

- Estimate parameters \( \theta = (\alpha, \beta, \gamma, \phi) \) and initial states \( x_0 = (\ell_0, b_0, s_0, s_{-1}, \ldots, s_{-m+1}) \) by minimizing \( L^* \).
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Parameter restrictions

**Usual region**

- Traditional restrictions in the methods $0 < \alpha, \beta^*, \gamma^*, \phi < 1$ — equations interpreted as weighted averages.
- In models we set $\beta = \alpha \beta^*$ and $\gamma = (1 - \alpha) \gamma^*$ therefore $0 < \alpha < 1$, $0 < \beta < \alpha$ and $0 < \gamma < 1 - \alpha$.
- $0.8 < \phi < 0.98$ — to prevent numerical difficulties.

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State space models 1: Exponential smoothing
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**Akaike’s Information Criterion**

\[ AIC = -2 \log(L) + 2k \]

where \( L \) is the likelihood and \( k \) is the number of parameters initial states estimated in the model.

**Corrected AIC**

\[ AIC_c = AIC + \frac{2(k + 1)(k + 2)}{T - k} \]

which is the AIC corrected (for small sample bias).

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From Hyndman et al. (IJF, 2002):

- Apply each model that is appropriate to the data. Optimize parameters and initial values using MLE (or some other criterion).
- Select best method using AIC:

\[
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where \( p = \# \) parameters.
- Produce forecasts using best method.
- Obtain prediction intervals using underlying state space model.

Method performed very well in M3 competition.
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Forecasting with ETS models

- Point forecasts obtained by iterating equations for \( t = T + 1, \ldots, T + h \), setting \( \varepsilon_t = 0 \) for \( t > T \).
- Not the same as \( \mathbb{E}(y_{t+h}|x_t) \) unless trend and seasonality are both additive.
- Point forecasts for ETS(A,x,y) are identical to ETS(M,x,y) if the parameters are the same.
- Prediction intervals will differ between models with additive and multiplicative methods.
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- Otherwise, simulate future sample paths, conditional on last estimate of states, and obtain PI from percentiles of simulated paths.
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For example, for ETS(M,A,N):

- \( y_{T+1} = (\ell_T + b_T)(1 + \varepsilon_{T+1}) \)
- Therefore \( \hat{y}_{T+1|T} = \ell_T + b_T \)

\[
y_{T+2} = (\ell_{T+1} + b_{T+1})(1 + \varepsilon_{T+1}) = (\ell_T + b_T)(1 + \alpha \varepsilon_{T+1}) + b_T + \beta(\ell_T + b_T)\varepsilon_{T+1})(1 + \varepsilon_{T+1})
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- Therefore \( \hat{y}_{T+2|T} = \ell_T + 2b_T \) and so on.

Identical forecast with Holt’s linear method and ETS(A,A,N). So the point forecasts obtained from the method and from the two models that underly the method are identical (assuming the same parameter values are used).
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- $y_{T+2} = (\ell_{T+1} + b_{T+1})(1 + \varepsilon_{T+1}) = [(\ell_T + b_T)(1 + \alpha \varepsilon_{T+1}) + b_T + \beta(\ell_T + b_T)\varepsilon_{T+1}] (1 + \varepsilon_{T+1})$
- Therefore $\hat{y}_{T+2|T} = \ell_T + 2b_T$ and so on.

Identical forecast with Holt’s linear method and ETS(A,A,N). So the point forecasts obtained from the method and from the two models that underly the method are identical (assuming the same parameter values are used).
Forecasting with ETS models

**Point forecasts:** iterate the equations for $t = T + 1, T + 2, \ldots, T + h$ and set all $\varepsilon_t = 0$ for $t > T$.

For example, for ETS(M,A,N):

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**Forecasting with ETS models**

**Prediction intervals:** cannot be generated using the methods.

- The prediction intervals will differ between models with additive and multiplicative methods.
- Exact formulae for some models.
- More general to simulate future sample paths, conditional on the last estimate of the states, and to obtain prediction intervals from the percentiles of these simulated future paths.
- Options are available in R using the `forecast` function in the forecast package.
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1 The state space perspective
2 Simple exponential smoothing
3 Trend methods
4 Seasonal methods
5 Taxonomy of exponential smoothing methods
6 Innovations state space models
7 ETS in R
Exponential smoothing

Forecasts from ETS(M,Md,M)

Year
Total scripts (millions)
1995 2000 2005 2010
0.4 0.6 0.8 1.0 1.2 1.4 1.6

State space models
1: Exponential smoothing 51
Exponential smoothing

Forecasts from ETS(M,Md,M)

Year
1995 2000 2005 2010
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library(forecast)
fit <- ets(h02)
fcast <- forecast(fit)
plot(fcast)
Exponential smoothing

```r
> fit
ETS(M,Md,M)

Smoothing parameters:
  alpha = 0.3318
  beta  = 4e-04
  gamma = 1e-04
  phi   = 0.9695
Initial states:
  l = 0.4003
  b = 1.0233
  s = 0.8575 0.8183 0.7559 0.7627 0.6873 1.2884
     1.3456 1.1867 1.1653 1.1033 1.0398 0.9893

sigma:  0.0651

<table>
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<th>AIC</th>
<th>AICc</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>-121.97999</td>
<td>-118.68967</td>
<td>-65.57195</td>
</tr>
</tbody>
</table>
```

State space models

1: Exponential smoothing
ets(y, model="ZZZ", damped=NULL, alpha=NULL, beta=NULL, gamma=NULL, phi=NULL, additive.only=FALSE, lambda=NULL, lower=c(rep(0.0001,3),0.80), upper=c(rep(0.9999,3),0.98), opt.crit=c("lik","amse","mse","sigma"), nmse=3, bounds=c("both","usual","admissible"), ic=c("aic","aicc","bic"), restrict=TRUE)
The `ets()` function in R

- **y**
  The time series to be forecast.

- **model**
  Use the ETS classification and notation: “N” for none, “A” for additive, “M” for multiplicative, or “Z” for automatic selection. Default `Z` all components are selected using the information criterion.

- **damped**
  If `damped=TRUE`, then a damped trend will be used (either `Ad` or `Md`). If `damped=FALSE` (the default), then either a damped or a non-damped trend will be selected according to the information criterion chosen.
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The `ets()` function in R

- **alpha, beta, gamma, phi**
  The values of the smoothing parameters can be specified using these arguments. If they are set to **NULL** (the default value for each of them), the parameters are estimated.

- **additive.only**
  Only models with additive components will be considered if `additive.only=TRUE`. Otherwise all models will be considered.

- **lambda**
  Box-Cox transformation parameter. It will be ignored if `lambda=NULL` (the default value). Otherwise, the time series will be transformed before the model is estimated. When `lambda` is not NULL, `additive.only` is set to **TRUE**.
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- **lower, upper** bounds for the parameter estimates of \( \alpha, \beta, \gamma \) and \( \phi \).
- **opt.crit=lik** (default) optimisation criterion used for estimation.
- **bounds** Constraints on the parameters.
  - usual region – "bounds=usual";
  - admissible region – "bounds=admissible";
  - "bounds=both" (the default) requires the parameters to satisfy both sets of constraints.
- **ic=aic** (the default) information criterion to be used in selecting models.
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t: forecast(object, 
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    level=c(80,95), fan=FALSE, 
    simulate=FALSE, bootstrap=FALSE, 
    npaths=5000, PI=TRUE, lambda=object$lambda, ..
```

- **object**: the object returned by the ets() function.
- **h**: the number of periods to be forecast.
- **level**: the confidence level for the prediction intervals.
- **fan**: if fan=TRUE, suitable for fan plots.
- **simulate**
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