Solutions to 6.7 Exercises

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Exercise 6.1

ETS(M,N,N)

(a) From equation (6.3),

$$\theta_j = \mu^2_{n+j|n} + \sigma^2 \sum_{i=1}^{j-1} c_i^2 \theta_{j-i}$$

where $c_i = w' D^{j-1} g = \alpha$ and $\mu^2_{n+j|n} = \ell^2_n$.

$$\begin{aligned}
\text{So} \qquad \theta_j &= \ell^2_n + \sigma^2 \sum_{i=1}^{j-1} \alpha^2 \theta_{j-i} \\
&= \ell^2_n + \alpha^2\sigma^2[\theta_{j-1} + \theta_{j-2} + \cdots + \theta_1] \\
&= \ell^2_n + \alpha^2\sigma^2 \left\{\ell^2_n + \alpha^2\sigma^2[\theta_{j-2} + \theta_{j-3} + \cdots + \theta_1] + \theta_{j-2} + \cdots + \theta_1\right\} \\
&= \ell^2_n (1 + \alpha^2\sigma^2)^{j-1}.
\end{aligned}$$

(b) From equation (6.2),

$$\begin{aligned}
v_{n+h|n} &= (1+\sigma^2)\theta_h - \mu^2_{n+h|n} \\
&= (1+\sigma^2)[\ell^2_n (1+\alpha^2\sigma^2)^{h-1}] - \ell^2_n \\
&= \ell^2_n \left\{[(1+\sigma^2)(1+\alpha^2\sigma^2)^{h-1}] - 1\right\}
\end{aligned}$$

Exercise 6.2

ETS(A,A,A). From equation (6.23):

$$\begin{aligned}
v_{n+h|n} &= \sigma^2 \left[1 + \sum_{j=1}^{h-1} (\alpha^2 + 2\alpha\beta j + \beta^2 j^2 + \left\{\gamma^2 + 2\alpha\gamma + 2\beta\gamma j\right\} d_{j,m})\right] \\
&= \sigma^2 \left[1 + (h-1)\alpha^2 + 2\alpha\beta \sum_{j=1}^{h-1} j + \beta^2 \sum_{j=1}^{h-1} j^2 + \sum_{j=1}^{h-1} (\gamma^2 + 2\alpha\gamma + 2\beta\gamma j) d_{j,m}\right] \\
&= \sigma^2 \left[1 + (h-1)\alpha^2 + \alpha\beta h(h-1) + \frac{\beta^2}{6} h(h-1)[2(h-1)+1] + \gamma^2 \sum_{j=1}^{h-1} d_{j,m} + \right. \\
&\qquad \left. + 2\alpha\gamma \sum_{j=1}^{h-1} d_{j,m} + 2\beta\gamma \sum_{j=1}^{h-1} j d_{j,m}\right] \\
&= \sigma^2 \left[1 + (h-1)\alpha^2 + \alpha\beta h(h-1) + \frac{\beta^2}{6} h(h-1)(2h-1) + \gamma^2 h_m + 2\alpha\gamma h_m + 2\beta\gamma \sum_{j=1}^{h-1} j d_{j,m}\right].
\end{aligned}$$

$$\text{Now} \qquad \sum_{j=1}^{h-1} j d_{j,m} = 0 \qquad \text{if} \qquad h-1 < m$$

$$\text{and} \qquad \sum_{j=1}^{h-1} j d_{j,m} = \sum_{\ell=1}^{hm} j_{\ell,m} = \frac{1}{2} m h_m (h_m+1) \qquad \text{if } h-1 \geq m.$$

Therefore

$$\begin{aligned} v_{n+h|n} &= \sigma^2 \left\{ 1 + (h-1)\left[\alpha^2 + \alpha\beta h + \frac{\beta^2}{6}(h-1)h(2h-1)\right] + \gamma^2 h_m + 2\alpha\gamma h_m + \beta\gamma h_m (h_m+1)m \right\} \\ &= \sigma^2 \left\{ 1 + (h-1)\left[\alpha^2 + \alpha\beta h + \frac{\beta^2}{6}(h-1)h(2h-1)\right] + \gamma h_m \left[\gamma + 2\alpha + \beta(h_m+1)m\right] \right\}. \end{aligned}$$

Exercise 6.3

The prediction interval for the ETS(A,Ad,N) model is given by $\hat{\mu}_{n+h|n} \pm z_{\alpha/2}\sqrt{v_{n+h|n}}$, where the forecast mean is given by (p81) $\hat{\mu}_{n+h|n} = \hat{\ell}_n + \hat{\phi}_h \hat{b}_n$ and the forecast variance is given by (p82)

$$v_{n+h|n} = \sigma^2 \left[1 + \alpha^2(h-1) + \frac{\beta\phi h}{(1-\phi)^2}\{2\alpha(1-\phi) + \beta\phi\} - \frac{\beta\phi(1-\phi)^h}{(1-\phi)^2(1-\phi^2)}\left\{2\alpha(1-\phi^2) + \beta\phi(1+2\phi-\phi^h)\right\} \right],$$

with $\phi_h = \phi + \phi^1 + \phi^2 + \cdots + \phi^h$. The parameters given on p28 are $\alpha = 0.99$, $\beta = 0.12$, $\phi = 0.8$, $\ell_0 = 5.3$, and $b_0 = 0.71$. However, σ^2 is not given and changes to the package mean that slightly different estimates are now obtained:

```
> fit <- ets(bonds, model="AAN", damped=TRUE)
> fit
ETS(A,Ad,N)

  Smoothing parameters:
    alpha = 0.9999
    beta  = 0.1608
    phi   = 0.8

  Initial states:
    l = 5.5163
    b = 0.2967

  sigma:  0.2394

# Intervals calculated using formula
> forecast(fit, h=12, level=95)
         Point Forecast    Lo 95    Hi 95
Jun 2004       4.791887 4.322616 5.261159
Jul 2004       4.865425 4.099731 5.631119
Aug 2004       4.924255 3.917714 5.930796
Sep 2004       4.971319 3.749872 6.192766
Oct 2004       5.008970 3.589651 6.428289
Nov 2004       5.039091 3.434922 6.643259
Dec 2004       5.063187 3.284927 6.841448
Jan 2005       5.082465 3.139374 7.025555
Feb 2005       5.097887 2.998130 7.197644
Mar 2005       5.110224 2.861095 7.359353
Apr 2005       5.120094 2.728169 7.512019
May 2005       5.127990 2.599230 7.656750
```

```
# Simulated intervals
> forecast(fit, simulate=TRUE, h=12, level=95)
         Point Forecast    Lo 95    Hi 95
Jun 2004       4.773503 4.307237 5.226732
Jul 2004       4.847040 4.109158 5.532933
Aug 2004       4.905870 3.944445 5.785264
Sep 2004       4.952934 3.809287 6.012705
Oct 2004       4.990585 3.650520 6.216636
Nov 2004       5.020706 3.505711 6.423038
Dec 2004       5.044803 3.364376 6.605925
Jan 2005       5.064080 3.263976 6.755975
Feb 2005       5.079502 3.073723 6.928992
Mar 2005       5.091840 2.944629 7.088639
Apr 2005       5.101710 2.814529 7.215576
May 2005       5.109606 2.695644 7.309356
```

Exercise 6.4

The prediction interval for the ETS(A,N,A) model is given by $\hat{\mu}_{n+h|n} \pm z_{\alpha/2}\sqrt{v_{n+h|n}}$, where the forecast mean is given by (p81) $\hat{\mu}_{n+h|n} = \hat{\ell}_n + \hat{s}_{n-m+h_4^+}$ and the forecast variance is given by (p82)

$$v_{n+h|n} = \sigma^2[1 + \alpha^2(h-1) + \gamma h_4(2\alpha + \gamma)]$$

with $h_4 = \lfloor h - 1/4 \rfloor$. The parameters given on p28 are $\alpha = 0.61$, $\gamma = 0.01$, $\ell_0 = 343.4$, $s_{-3} = 24.99$, $s_{-2} = 21.40$, $s_{-} = -44.96$, and $s_0 = -1.42$. However, σ^2 is not given and changes to the package mean that slightly different estimates are now obtained:

```
> fit <- ets(ukcars, model="ANA")

> fit
ETS(A,N,A)

  Smoothing parameters:
    alpha = 0.6267
    gamma = 2e-04

  Initial states:
    l = 338.4757
    s=-0.5313 -45.3246 20.6084 25.2476

  sigma:  25.3264

# Intervals calculated using formula
> forecast(fit, h=12, level=95)
        Point Forecast    Lo 95    Hi 95
2005 Q2       426.8056 377.1667 476.4444
2005 Q3       360.8705 302.2883 419.4527
2005 Q4       405.6569 339.3219 471.9918
2006 Q1       431.4437 358.1757 504.7116
2006 Q2       426.8056 347.2063 506.4048
2006 Q3       360.8705 275.4076 446.3334
2006 Q4       405.6569 314.7043 496.6094
2007 Q1       431.4437 335.3176 527.5697
2007 Q2       426.8056 325.7705 527.8406
2007 Q3       360.8705 255.1542 466.5868
2007 Q4       405.6569 295.4553 515.8585
2008 Q1       431.4437 316.9349 545.9524
```

```
# Simulated intervals
> forecast(fit, simulate=TRUE, h=12, level=95)
        Point Forecast    Lo 95    Hi 95
2005 Q2       426.8056 377.4023 476.3763
2005 Q3       360.8705 303.5556 420.8765
2005 Q4       405.6569 340.3205 472.5073
2006 Q1       431.4437 358.2238 504.7932
2006 Q2       426.8056 349.0306 508.9335
2006 Q3       360.8705 275.7249 447.8884
2006 Q4       405.6569 314.3210 497.9019
2007 Q1       431.4437 335.1558 530.3091
2007 Q2       426.8056 327.5703 529.4180
2007 Q3       360.8705 251.3072 466.9184
2007 Q4       405.6569 296.1100 518.1099
2008 Q1       431.4437 317.1493 546.6531
```