Solutions to 4.5 Exercises

Facultad de Ciencias Econmicas y Estadistica
Universidad Nacional de Rosario, Argentina
M.T.Blaconá - L. Magnano - L. Andreozzi

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Exercise 4.1

Additive Model ETS(A,N,N)

$$
\begin{aligned}
y_t &= x_{t-1} + \varepsilon_t \\
x_t &= x_{t-1} + \alpha\varepsilon_t
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\mathrm{V}(y_1 \mid x_0) &= \mathrm{V}(x_0 + \varepsilon_1 \mid x_0) = \sigma_A^2 \\
\mathrm{V}(y_2 \mid x_0) &= \mathrm{V}(x_1 + \varepsilon_2 \mid x_0) = \mathrm{V}(x_0 + \alpha\varepsilon_1 + \varepsilon_2 \mid x_0) \\
&= \alpha^2\sigma_A^2 + \sigma_A^2 = (1+\alpha^2)\sigma_A^2 \\
\mathrm{V}(y_3 \mid x_0) &= \mathrm{V}(x_2 + \varepsilon_3 \mid x_0) = \mathrm{V}(x_1 + \alpha\varepsilon_2 + \varepsilon_3 \mid x_0) \\
&= \mathrm{V}(x_0 + \alpha\varepsilon_1 + \alpha\varepsilon_2 + \varepsilon_3 \mid x_0) \\
&= \alpha^2\sigma_A^2 + \alpha^2\sigma_A^2 + \sigma_A^2 = (1+2\alpha^2)\sigma_A^2 \\
&\vdots \\
\mathrm{V}(y_t \mid x_0) &= \mathrm{V}(x_{t-1} + \varepsilon_t \mid x_0) = [1 + (t-1)\alpha^2]\sigma_A^2
\end{aligned}
$$

Multiplicative Model ETS(M,N,N)

$$
\begin{aligned}
y_t &= x_{t-1}(1 + \varepsilon_t) \\
x_t &= x_{t-1}(1 + \alpha\varepsilon_t)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{V}(y_1 \mid x_0) &= \mathrm{V}\left[x_0(1+\varepsilon_1) \mid x_0\right] \\
&= x_0^2\sigma_M^2 \\
\mathrm{V}(y_2 \mid x_0) &= \mathrm{V}\left[x_1(1+\varepsilon_2) \mid x_0\right] \\
&= \mathrm{V}[x_0(1+\alpha\varepsilon_1)(1+\varepsilon_2) \mid x_0] \\
&= x_0^2(\alpha^2\sigma_M^2 + \sigma_M^2 + \alpha^2\sigma_M^2\sigma_M^2) = x_0^2[\alpha^2\sigma_M^2(1+\sigma_M^2) + \sigma_M^2] \\
&= x_0^2[(1+\alpha^2\sigma_M^2)(1+\sigma_M^2) - 1] \\
\mathrm{V}(y_3 \mid x_0) &= \mathrm{V}[x_2(1+\varepsilon_3) \mid x_0] \\
&= \mathrm{V}[x_1(1+\alpha\varepsilon_2)(1+\varepsilon_3) \mid x_0] \\
&= \mathrm{V}[x_0(1+\alpha\varepsilon_1)(1+\alpha\varepsilon_2)(1+\varepsilon_3) \mid x_0] \\
&= x_0^2\mathrm{V}(\alpha\varepsilon_1 + \alpha\varepsilon_2 + \varepsilon_3 + \alpha\varepsilon_1\varepsilon_3 + \alpha\varepsilon_2\varepsilon_3 + \alpha^2\varepsilon_1\varepsilon_2 + \alpha^2\varepsilon_1\varepsilon_2\varepsilon_3 + 1) \\
&= x_0^2[\alpha^2\sigma^2 + \alpha^2\sigma_M^2 + \sigma_M^2 + \alpha^2\sigma_M^2\sigma_M^2 + \alpha^2\sigma_M^2\sigma_M^2 + \alpha^4\sigma_M^2\sigma_M^2 + \alpha^4\sigma_M^2\sigma_M^2\sigma_M^2] \\
&= x_0^2[(1+\sigma^2)(1+\alpha^2\sigma_M^2)] \\
&\vdots \\
\mathrm{V}(y_t \mid x_0) &= x_0^2\left[(1+\sigma_M^2)(1+\alpha^2\sigma_M^2)^{t-1} - 1\right]
\end{aligned}
$$

Exercise 4.2

ETS(A,M,M)

$$\begin{aligned} y_t &= \ell_{t-1}b_{t-1}s_{t-m} + \varepsilon_t \\ \ell_t &= \ell_{t-1}b_{t-1} + \alpha\epsilon_t/s_{t-m} \\ b_t &= b_{t-1} + \beta\varepsilon/(s_{t-m}\ell_{t-1}) \\ s_t &= s_{t-m} + \gamma\varepsilon_t/(\ell_{t-1}b_{t-1}) \end{aligned}$$

In vector form:

$$\begin{bmatrix} \ell_t \\ b_t \\ s_t \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} = \begin{bmatrix} \ell_{t-1}b_{t-1} \\ b_{t-1} \\ s_{t-m} \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} - \begin{bmatrix} \alpha/s_{t-m} \\ \beta/(s_{t-m}\ell_{t-1}) \\ \gamma/(\ell_{t-1}b_{t-1}) \\ 0 \\ \vdots \\ 0 \end{bmatrix} (\ell_{t-1}b_{t-1}s_{t-m}) + \begin{bmatrix} \alpha/s_{t-m} \\ \beta/(s_{t-m}\ell_{t-1}) \\ \gamma/(\ell_{t-1}b_{t-1}) \\ 0 \\ \vdots \\ 0 \end{bmatrix} y_t,$$

or

$$\boldsymbol{x}_t = \left[\boldsymbol{f}(x_{t-1}) - \boldsymbol{g}(x_{t-1})\frac{\boldsymbol{w}(x_{t-1})}{\boldsymbol{r}(x_{t-1})}\right] + \frac{\boldsymbol{g}(x_{t-1})}{\boldsymbol{r}(x_{t-1})}y_t,$$

where

$$\boldsymbol{x}_t = \begin{bmatrix} \ell_t \\ b_t \\ s_t \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} \qquad \boldsymbol{g}(x_{t-1}) = \begin{bmatrix} \alpha/s_{t-m} \\ \beta/(s_{t-m}\ell_{t-1}) \\ \gamma/(\ell_{t-1}b_{t-1}) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \boldsymbol{f}(x_{t-1}) = \begin{bmatrix} \ell_{t-1}b_{t-1} \\ b_{t-1} \\ s_{t-m} \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix}$$

$$\boldsymbol{w}(x_{t-1}) = (\ell_{t-1}b_{t-1}s_{t-m}) \qquad \text{and} \qquad \boldsymbol{r}(x_{t-1}) = 1.$$

Thus the recursive relationships are given by:

$$\begin{aligned} \ell_t &= \ell_{t-1}b_{t-1} - \alpha\ell_{t-1}b_{t-1} + \alpha s_{t-m}/y_t \\ &= (1-\alpha)\ell_{t-1}b_{t-1} + \alpha s_{t-m}/y_t \\ b_t &= b_{t-1} - \beta b_{t-1} + \beta y_t/(s_{t-m}\ell_{t-1}) \\ &= b_{t-1} - \beta b_{t-1} + (\beta/\alpha)\left[\ell_t - (1-\alpha)\ell_{t-1}b_{t-1}\right]/\ell_{t-1} \\ &= b_{t-1} - \beta b_{t-1} + (\beta/\alpha)\left[\ell_t - \ell_{t-1}b_{t-1} + \alpha\ell_{t-1}b_{t-1}\right]/\ell_{t-1} \\ &= b_{t-1} - \beta b_{t-1} + (\beta/\alpha) + \ell_t/\ell_{t-1} + (\beta/\alpha)b_{t-1} + \beta b_{t-1} \\ &= (\beta/\alpha)\ell_t/\ell_{t-1} + (1-\beta/\alpha)b_{t-1} \\ s_t &= s_{t-m} - \gamma s_{t-m} + \gamma y_t/(\ell_{t-1}b_{t-1}) \\ &= (1-\gamma)s_{t-m} + \gamma y_t/(\ell_{t-1}b_{t-1}). \end{aligned}$$

ETS(A,Md,M)

$$\begin{aligned} y_t &= \ell_{t-1}b_{t-1}^{\phi}s_{t-m} + \varepsilon_t \\ \ell_t &= \ell_{t-1}b_{t-1}^{\phi} + \alpha\epsilon_t/s_{t-m} \\ b_t &= b_{t-1}^{\phi} + \beta\varepsilon_t/(s_{t-m}\ell_{t-1}) \\ s_t &= s_{t-m} + \gamma\varepsilon_t/(\ell_{t-1}b_{t-1}^{\phi}) \end{aligned}$$

In vector form:

$$\begin{bmatrix} \ell_t \\ b_t \\ s_t \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} = \begin{bmatrix} \ell_{t-1}b_{t-1}^{\phi} \\ b_{t-1}^{\phi} \\ s_{t-m} \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} - \begin{bmatrix} \alpha/s_{t-m} \\ \beta/(s_{t-m}\ell_{t-1}) \\ \gamma/(\ell_{t-1}b_{t-1}^{\phi}) \\ 0 \\ \vdots \\ 0 \end{bmatrix} (\ell_{t-1}b_{t-1}^{\phi}s_{t-m}) + \begin{bmatrix} \alpha/s_{t-m} \\ \beta/(s_{t-m}\ell_{t-1}) \\ \gamma/(\ell_{t-1}b_{t-1}^{\phi}) \\ 0 \\ \vdots \\ 0 \end{bmatrix} y_t,$$

or

$$\boldsymbol{x}_t = \left[\boldsymbol{f}(x_{t-1}) - \boldsymbol{g}(x_{t-1})\frac{\boldsymbol{w}(x_{t-1})}{\boldsymbol{r}(x_{t-1})}\right] + \frac{\boldsymbol{g}(x_{t-1})}{\boldsymbol{r}(x_{t-1})}y_t,$$

where

$$\boldsymbol{x}_t = \begin{bmatrix} \ell_t \\ b_t \\ s_t \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} \qquad \boldsymbol{g}(x_{t-1}) = \begin{bmatrix} \alpha/s_{t-m} \\ \beta/(s_{t-m}\ell_{t-1}) \\ \gamma/(\ell_{t-1}b_{t-1}) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \boldsymbol{f}(x_{t-1}) = \begin{bmatrix} \ell_{t-1}b_{t-1}^{\phi} \\ b_{t-1}^{\phi} \\ s_{t-m} \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix}$$

$$\boldsymbol{w}(x_{t-1}) = (\ell_{t-1}b_{t-1}^{\phi}s_{t-m}) \qquad \text{and} \qquad \boldsymbol{r}(x_{t-1}) = 1.$$

Thus the recursive relationships are given by:

$$\begin{aligned}
\ell_t &= \ell_{t-1}b_{t-1}^{\phi} - \alpha\ell_{t-1}b_{t-1}^{\phi} + \alpha s_{t-m}/y_t \\
&= (1-\alpha)\ell_{t-1}b_{t-1}^{\phi} + \alpha s_{t-m}/y_t \\
b_t &= b_{t-1}^{\phi} - \beta b_{t-1}^{\phi} + \beta y_t/(s_{t-m}\ell_{t-1}) \\
&= b_{t-1}^{\phi} - \beta b_{t-1}^{\phi} + (\beta/\alpha)\left[\ell_t - (1-\alpha)\ell_{t-1}b_{t-1}^{\phi}\right]/\ell_{t-1} \\
&= b_{t-1}^{\phi} - \beta b_{t-1}^{\phi} + (\beta/\alpha)\left[\ell_t - \ell_{t-1}b_{t-1}^{\phi} + \alpha\ell_{t-1}b_{t-1}^{\phi}\right]/\ell_{t-1} \\
&= b_{t-1}^{\phi} - \beta b_{t-1}^{\phi} + (\beta/\alpha) + \ell_t/\ell_{t-1} + (\beta/\alpha)b_{t-1}^{\phi} + \beta b_{t-1}^{\phi} \\
&= (\beta/\alpha)\ell_t/\ell_{t-1} + (1-\beta/\alpha)b_{t-1}^{\phi} \\
s_t &= s_{t-m} - \gamma s_{t-m} + \gamma y_t/(\ell_{t-1}b_{t-1}^{\phi}) \\
&= (1-\gamma)s_{t-m} + \gamma y_t/(\ell_{t-1}b_{t-1}^{\phi}).
\end{aligned}$$

Exercise 4.3

ETS(M,M,M)

$$\begin{aligned}
y_t &= (\ell_{t-1}b_{t-1}s_{t-m})(1+\varepsilon_t) \\
\ell_t &= (\ell_{t-1}b_{t-1})(1+\alpha\epsilon_t) \\
b_t &= b_{t-1}(1+\beta\varepsilon_t) \\
s_t &= s_{t-m}(1+\gamma\varepsilon_t)
\end{aligned}$$

In vector form:

$$\begin{bmatrix} \ell_t \\ b_t \\ s_t \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} = \begin{bmatrix} \ell_{t-1}b_{t-1} \\ b_{t-1} \\ s_{t-m} \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} - \begin{bmatrix} \alpha(\ell_{t-1}b_{t-1}) \\ \beta b_{t-1} \\ \gamma s_{t-m} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha/s_{t-m} \\ \beta/(s_{t-m}\ell_{t-1}) \\ \gamma/(\ell_{t-1}b_{t-1}) \\ 0 \\ \vdots \\ 0 \end{bmatrix} y_t,$$

or

$$\boldsymbol{x}_t = \left[\boldsymbol{f}(x_{t-1}) - \boldsymbol{g}(x_{t-1})\frac{\boldsymbol{w}(x_{t-1})}{\boldsymbol{r}(x_{t-1})}\right] + \frac{\boldsymbol{g}(x_{t-1})}{\boldsymbol{r}(x_{t-1})}y_t$$

where

$$\boldsymbol{x}_t = \begin{bmatrix} \ell_t \\ b_t \\ s_t \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} \qquad \boldsymbol{g}(x_{t-1}) = \begin{bmatrix} \alpha\ell_{t-1}b_{t-1} \\ \beta b_{t-1} \\ \gamma s_{t-m} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \boldsymbol{f}(x_{t-1}) = \begin{bmatrix} \ell_{t-1}b_{t-1} \\ b_{t-1} \\ s_{t-m} \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix}$$

$$\boldsymbol{w}(x_{t-1}) = (\ell_{t-1}b_{t-1}s_{t-m}) \qquad \text{and} \qquad \boldsymbol{r}(x_{t-1}) = (\ell_{t-1}b_{t-1}s_{t-m}).$$

Thus the recursive relationships are given by:

$$\begin{aligned}
\ell_t &= \ell_{t-1}b_{t-1} - \alpha\ell_{t-1}b_{t-1} + \alpha y_t/s_{t-m} \\
&= (1-\alpha)\ell_{t-1}b_{t-1} + \alpha y_t/s_{t-m} \\
b_t &= b_{t-1} - \beta b_{t-1} + \beta y_t/(s_{t-m}\ell_{t-1}) \\
&= b_{t-1} - \beta b_{t-1} + (\beta/\alpha)\left[\ell_t - (1-\alpha)\ell_{t-1}b_{t-1}\right]/\ell_{t-1} \\
&= b_{t-1} - \beta b_{t-1} + (\beta/\alpha)\left[\ell_t - \ell_{t-1}b_{t-1} + \alpha\ell_{t-1}b_{t-1}\right]/\ell_{t-1} \\
&= b_{t-1} - \beta b_{t-1} + \beta/\alpha + \ell_t/\ell_{t-1} + (\beta/\alpha)b_{t-1} + \beta b_{t-1} \\
&= (\beta/\alpha)\ell_t/\ell_{t-1} + (1-\beta/\alpha)b_{t-1} \\
s_t &= s_{t-m} - \gamma s_{t-m} + \gamma y_t/(\ell_{t-1}b_{t-1}) \\
&= (1-\gamma)s_{t-m} + \gamma y_t/(\ell_{t-1}b_{t-1})
\end{aligned}$$

ETS(M,Md,M)

$$\begin{aligned}
y_t &= (\ell_{t-1}b_{t-1}^{\phi}s_{t-m})(1+\varepsilon_t) \\
\ell_t &= (\ell_{t-1}b_{t-1}^{\phi})(1+\alpha\epsilon_t) \\
b_t &= b_{t-1}^{\phi}(1+\beta\varepsilon_t) \\
s_t &= s_{t-m}(1+\gamma\varepsilon_t)
\end{aligned}$$

In vector form:

$$\begin{bmatrix} \ell_t \\ b_t \\ s_t \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} = \begin{bmatrix} \ell_{t-1}b_{t-1}^{\phi} \\ b_{t-1}^{\phi} \\ s_t \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} - \begin{bmatrix} \alpha(\ell_{t-1}b_{t-1}^{\phi}) \\ \beta b_{t-1}^{\phi} \\ \gamma s_{t-m} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha/s_{t-m} \\ \beta/(s_{t-m}\ell_{t-1}) \\ \gamma/(\ell_{t-1}b_{t-1}^{\phi}) \\ 0 \\ \vdots \\ 0 \end{bmatrix} y_t$$

or

$$\boldsymbol{x}_t = \left[\boldsymbol{f}(x_{t-1}) - \boldsymbol{g}(x_{t-1})\frac{\boldsymbol{w}(x_{t-1})}{\boldsymbol{r}(x_{t-1})}\right] + \frac{\boldsymbol{g}(x_{t-1})}{\boldsymbol{r}(x_{t-1})}y_t$$

where

$$\boldsymbol{x}_t = \begin{bmatrix} \ell_t \\ b_t \\ s_t \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} \qquad \boldsymbol{g}(x_{t-1}) = \begin{bmatrix} \alpha(\ell_{t-1}b_{t-1}^{\phi}) \\ \beta b_{t-1}^{\phi} \\ \gamma s_{t-m} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \boldsymbol{f}(x_{t-1}) = \begin{bmatrix} \ell_{t-1}b_{t-1}^{\phi} \\ b_{t-1}^{\phi} \\ s_t \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix}$$

$$\boldsymbol{w}(x_{t-1}) = (\ell_{t-1}b_{t-1}^{\phi}s_{t-m}) \qquad \text{and} \qquad \boldsymbol{r}(x_{t-1}) = (\ell_{t-1}b_{t-1}^{\phi}s_{t-m}).$$

Thus the recursive relationships are given by:

$$\begin{aligned}
\ell_t &= \ell_{t-1}b_{t-1}^{\phi} - \alpha\ell_{t-1}b_{t-1}^{\phi} + \alpha y_t/s_{t-m} \\
&= (1-\alpha)\ell_{t-1}b_{t-1}^{\phi} + \alpha y_t/s_{t-m} \\
b_t &= b_{t-1}^{\phi} - \beta b_{t-1}^{\phi} + \beta y_t/(s_{t-m}\ell_{t-1}) \\
&= b_{t-1}^{\phi} - \beta b_{t-1}^{\phi} + (\beta/\alpha)\left[\ell_t - (1-\alpha)\ell_{t-1}b_{t-1}^{\phi}\right]/\ell_{t-1} \\
&= b_{t-1}^{\phi} - \beta b_{t-1}^{\phi} + (\beta/\alpha)\left[\ell_t - \ell_{t-1}b_{t-1}^{\phi} + \alpha\ell_{t-1}b_{t-1}^{\phi}\right]/\ell_{t-1} \\
&= b_{t-1}^{\phi} - \beta b_{t-1}^{\phi} + \beta/\alpha + \ell_t/\ell_{t-1} + (\beta/\alpha)b_{t-1}^{\phi} + \beta b_{t-1}^{\phi} \\
&= (\beta/\alpha)\ell_t/\ell_{t-1} + (1-\beta/\alpha)b_{t-1}^{\phi} \\
s_t &= s_{t-m} - \gamma s_{t-m} + \gamma y_t/(\ell_{t-1}b_{t-1}^{\phi}) \\
&= (1-\gamma)s_{t-m} + \gamma y_t/(\ell_{t-1}b_{t-1}^{\phi})
\end{aligned}$$

Exercise 4.4

Local trend model

$$\begin{aligned}
y_t &= (\ell_{t-1} + b_{t-1})(1+\varepsilon_t) \\
\ell_t &= (\ell_{t-1} + b_{t-1})(1+\alpha\varepsilon_t) \\
b_t &= b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t
\end{aligned}$$

$$\begin{aligned}
\hat{y}_{t+1|t} &= \mathrm{E}[(\ell_t + b_t)(1+\varepsilon_{t+1}) \mid b_t, \ell_t] = \ell_t + b_t \\
e_{t+1|t} &= y_{t+1} - \hat{y}_{t+1|t} = (\ell_t + b_t)\varepsilon_t \\
\mathrm{V}(e_{t+1|t}) &= (\ell_t + b_t)^2\sigma^2
\end{aligned}$$

$$\begin{aligned}
\hat{y}_{t+2|t} &= \mathrm{E}[(\ell_{t+1} + b_{t+1})(1+\varepsilon_{t+2}) \mid b_t, \ell_t] \\
&= \mathrm{E}\left\{[(\ell_t + b_t)(1+\alpha\varepsilon_{t+2}) + (b_t + \beta(\ell_t + b_t)\varepsilon_{t+1})(1+\varepsilon_{t+2})] \mid b_t, \ell_t\right\} \\
&= \mathrm{E}\left\{[\ell_t + \alpha\varepsilon_{t+1} + b_t + \alpha b_t\varepsilon_{t+1} + b_t + b_t\varepsilon_{t+2} + \beta\ell_t\varepsilon_{t+1} + \beta\ell_t\varepsilon_{t+1}\varepsilon_{t+2} + \beta b_t\varepsilon_{t+1}\varepsilon_{t+2}] \mid b_t, \ell_t\right\} \\
&= \ell_t + b_t + b_t = \ell_t + 2b_t \\
e_{t+2|t} &= y_{t+2} - \hat{y}_{t+2|t} = \alpha\ell_t\varepsilon_{t+1} + \alpha b_t\varepsilon_{t+1} + b_t\varepsilon_{t+2} + \beta\ell_t\varepsilon_{t+1} + \beta\ell_t\varepsilon_{t+1}\varepsilon_{t+2} + \beta b_t\varepsilon_{t+1}\varepsilon_{t+2} \\
&= (\alpha\ell_t + \alpha b_t + \beta\ell_t)\varepsilon_{t+1} + b_t\varepsilon_{t+2} + (\beta\ell_t + \beta b_t)\varepsilon_{t+1}\varepsilon_{t+2} \\
\mathrm{V}(e_{t+2|t}) &= (\alpha\ell_t + \alpha b_t + \beta\ell_t)^2\sigma^2 + b_t^2\sigma^2 + (\beta\ell_t + \beta b_t)^2\sigma^2\sigma^2 \\
&= [(\alpha\ell_t + \alpha b_t + \beta\ell_t)^2 + b_t^2 + (\beta\ell_t + \beta b_t)^2\sigma^2]\sigma^2
\end{aligned}$$

Local Level Model with Drift

$$\begin{aligned}
y_t &= (\ell_{t-1} + b)(1+\varepsilon_t) \\
\ell_t &= (\ell_{t-1} + b)(1+\alpha\varepsilon_t)
\end{aligned}$$

$$\begin{aligned}
\hat{y}_{t+1|t} &= \mathrm{E}[(\ell_t + b)(1+\varepsilon_{t+1}) \mid y_t] = \ell_t + b \\
e_{t+1|t} &= y_{t+1} - \hat{y}_{t+1|t} = (\ell_{t-1} + b)\varepsilon_{t+1} \\
\mathrm{E}[e_{t+1|t}]^2 &= (\ell_{t-1} + b)^2\sigma^2 \\
\hat{y}_{t+2|t} &= \mathrm{E}[(\ell_{t+1} + b)(1+\varepsilon_{t+2}) \mid y_t] = \mathrm{E}\left\{[(\ell_t + b)(1+\alpha\varepsilon_{t+1}) + b](1+\varepsilon_{t+2}) \mid y_t\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{E}\left\{[(\ell_t + b) + (\ell_t + b)\alpha\varepsilon_{t+1} + b](1 + \varepsilon_{t+2}) \mid y_t\right\} \\
&= \mathrm{E}[(\ell_t + 2b) + (\ell_t + b)\alpha\varepsilon_{t+1} + (\ell_t + 2b)\varepsilon_{t+2} + (\ell_t + b)\alpha\varepsilon_{t+1}\varepsilon_{t+2}] = \ell_t + 2b \\
e_{t+2|t} &= y_{t+2} - \hat{y}_{t+2|t} \\
&= \alpha(\ell_t + b)\varepsilon_{t+1} + (\ell_t + 2b)\varepsilon_{t+2} + (\ell_t + b)\alpha\varepsilon_{t+1}\varepsilon_{t+2} \\
\mathrm{E}[e_{t+2|t}]^2 &= \alpha^2(\ell_t + b)^2\sigma^2 + (\ell_t + 2b)^2\sigma^2 + (\ell_t + b)^2\alpha^2\sigma^2\sigma^2 \\
&= \alpha^2(\ell_t + b)^2\sigma^2(1 + \sigma^2) + (\ell_t + 2b)^2\sigma^2
\end{aligned}
$$

$$
\begin{aligned}
\hat{y}_{t+3|t} &= \mathrm{E}[(\ell_{t+2} + b)(1 + \varepsilon_{t+3}) \mid y_t] \\
&= \mathrm{E}\left\{[(\ell_{t+1} + b)(1 + \alpha\varepsilon_{t+2}) + b](1 + \varepsilon_{t+3}) \mid y_t\right\} \\
&= E\left\{[[(\ell_t + b)(1 + \alpha\varepsilon_{t+1}) + b](1 + \alpha\varepsilon_{t+2}) + b](1 + \varepsilon_{t+3}) \mid y_t\right\} \\
&= \ell_t + b + b + b = \ell + 3b \\
e_{t+3|t} &= y_{t+3} - \hat{y}_{t+3|t} \\
&= (\ell_t + b)\alpha\varepsilon_{t+1} + (\ell_t + 2b)\alpha\varepsilon_{t+2} + (\ell_t + b)\alpha^2\varepsilon_{t+1}\varepsilon_{t+2} + (\ell_t + 3b)\varepsilon_{t+3} \\
&+ (\ell_t + b)\alpha\varepsilon_{t+1}\varepsilon_{t+3} + (\ell_t + 2b)\alpha\varepsilon_{t+2}\varepsilon_{t+3} + (\ell_t + b)^2\varepsilon_{t+1}\varepsilon_{t+2}\varepsilon_{t+3} \\
\mathrm{E}[e_{t+3|t}]^2 &= (\ell_t + b)^2\alpha^2\sigma^2 + (\ell_t + 2b)^2\alpha^2\sigma^2 + (\ell_t + b)^2\alpha^4\sigma^2\sigma^2 + (\ell_t + 3b)^2\sigma^2 \\
&+ (\ell_t + b)^2\alpha^2\sigma^2\sigma^2 + (\ell_t + 2b)^2\alpha^2\sigma^2\sigma^2 + (\ell_t + b)^2\alpha^4\sigma^2\sigma^2\sigma^2 \\
&= (\ell_t + b)^2\alpha^2\sigma^2[1 + 2\alpha^2\sigma^2 + \alpha^2\sigma^4] + (\ell_t + 2b)^2\sigma^2\alpha^2(1 + \sigma^2) + (\ell_t + 3b)^2\sigma^2
\end{aligned}
$$

Exercise 4.5

Damped trend model

$$
\begin{aligned}
y_t &= (\ell_{t-1} + \phi b_{t-1})(1 + \varepsilon_t) \\
\ell_t &= (\ell_{t-1} + \phi b_{t-1}) + \alpha\varepsilon_t \\
b_t &= \phi b_{t-1} + \beta(\ell_{t-1} + \phi b_{t-1})\varepsilon_t
\end{aligned}
$$

$$
\begin{aligned}
\hat{y}_{t+1|t} &= \mathrm{E}[(\ell_t + \phi b_t)(1 + \varepsilon_{t+1}) \mid y_t] = \ell_t + \phi b_t \\
e_{t+1|t} &= y_{t+1} - \hat{y}_{t+1|t} = (\ell_t + \phi b_t)\varepsilon_t \\
\mathrm{V}(e_{t+1|t}) &= (\ell_t + \phi b_t)^2\sigma^2
\end{aligned}
$$

$$
\begin{aligned}
\hat{y}_{t+2|t} &= \mathrm{E}[(\ell_{t+1} + \phi b_{t+1})(1 + \varepsilon_{t+2}) \mid y_t] \\
&= \mathrm{E}\left\{[(\ell_t + \phi b_t)(1 + \alpha\varepsilon_{t+2}) + \phi(\phi b_t + \beta(\ell_t + \phi b_t)\varepsilon_{t+1})(1 + \varepsilon_{t+2})] \mid y_t\right\} \\
&= \mathrm{E}\left\{\left[\ell_t + \alpha\varepsilon_{t+1} + \phi b_t + \phi\alpha b_t\varepsilon_{t+1} + \phi^2 b_t + \phi^2 b_t\varepsilon_{t+2} + \beta\ell_t\varepsilon_{t+1} + \beta\ell_t\varepsilon_{t+1}\varepsilon_{t+2} + \beta\phi b_t\varepsilon_{t+1}\varepsilon_{t+2}\right] \mid y_t\right\} \\
&= \ell_t + \phi b_t + \phi^2 b_t = \ell_t + \phi b(1 + \phi) \\
e_{t+2|t} &= y_{t+2} - \hat{y}_{t+2|t} = \alpha\ell_t\varepsilon_{t+1} + \phi\alpha b_t\varepsilon_{t+1} + \phi^2 b_t\varepsilon_{t+2} + \beta\ell_t\varepsilon_{t+1} + \beta\ell_t\varepsilon_{t+1}\varepsilon_{t+2} + \beta b_t\varepsilon_{t+1}\varepsilon_{t+2} = \\
&= (\alpha\ell_t + \phi\alpha b_t + \beta\ell_t)\varepsilon_{t+1} + \phi^2 b_t\varepsilon_{t+2} + (\beta\ell_t + \beta\phi b_t)\varepsilon_{t+1}\varepsilon_{t+2} \\
\mathrm{V}(e_{t+2|t}) &= (\alpha\ell_t + \phi\alpha b_t + \beta\ell_t)^2\sigma^2 + \phi^4 b_t^2\sigma^2 + (\beta\ell_t + \beta\phi b_t)^2\sigma^2\sigma^2 \\
&= [(\alpha\ell_t + \phi\alpha b_t + \beta\ell_t)^2 + \phi^4 b_t^2 + (\beta\ell_t + \beta\phi b_t)^2\sigma^2]\sigma^2
\end{aligned}
$$

Exercise 4.6

The ETS(M,A,N) model is given by

$$
\begin{aligned}
y_t &= (\ell_{t-1} + b_{t-1})(1 + \varepsilon_t) \\
\ell_t &= (\ell_{t-1} + b_{t-1})(1 + \alpha\epsilon_t) \\
b_t &= b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t
\end{aligned}
$$

$$
\boldsymbol{x}_t = [\ell_t, b_t]', \qquad w(\boldsymbol{x}_{t-1}) = \ell_{t-1} + b_{t-1}, \qquad r(\boldsymbol{x}_{t-1}) = \ell_{t-1} + b_{t-1},
$$

$\boldsymbol{f}(\boldsymbol{x}_{t-1}) = [\ell_{t-1} + b_{t-1}, b_{t-1}]', \quad \boldsymbol{g}(\boldsymbol{x}_{t-1}) = [\alpha(\ell_{t-1} + b_{t-1}), \beta(\ell_{t-1} + b_{t-1})]', \quad$ and

$$\begin{aligned}
\boldsymbol{D} &= \boldsymbol{f}(\boldsymbol{x}_t) - \boldsymbol{g}(\boldsymbol{x}_t)w(\boldsymbol{x}_t)/r(\boldsymbol{x}_t) \\
&= \begin{bmatrix} \ell_t + b_t \\ b_t \end{bmatrix} - \begin{bmatrix} \alpha(\ell_t + b_t) \\ \beta(\ell_t + b_t) \end{bmatrix} \\
&= \begin{bmatrix} (\ell_t + b_t) - \alpha(\ell_t + b_t) \\ b_t - \beta(\ell_t + b_t) \end{bmatrix} \\
&= \begin{bmatrix} (1-\alpha)(\ell_t + b_t) \\ -\beta\ell_t + (1-\beta)b_t \end{bmatrix} \\
&= \begin{bmatrix} (1-\alpha) & (1-\alpha) \\ -\beta & (1-\beta) \end{bmatrix} \begin{bmatrix} \ell_{t-1} \\ b_{t-1} \end{bmatrix}
\end{aligned}$$

Eigenvalues

$$\begin{aligned}
\boldsymbol{D} - \boldsymbol{I}\lambda &= \begin{bmatrix} (1-\alpha) & (1-\alpha) \\ -\beta & (1-\beta) \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} (1-\alpha) - \lambda & (1-\alpha) \\ -\beta & (1-\beta) - \lambda \end{bmatrix} \\
&= [(1-\alpha) - \lambda][1 - \beta - \lambda] + \beta(1-\alpha) = 0
\end{aligned}$$

$$\lambda = \frac{1}{2}\left(2 - \alpha - \beta \pm \sqrt{(\alpha+\beta)^2 + 4\beta)}\right)$$

So $|\lambda| < 1$ iff $\alpha > 0$ and $0 < \beta < 4 - 2\alpha$.

Exercise 4.7

```
require(expsmooth)
plot(djiclose)
x <- window(djiclose[,"close"],start=1980)
fit <- forecast(ets(x,"MAN"),h=50)
fit2 <- rwf(x,drift=TRUE,h=50)
par(mfrow=c(2,2))
plot(fit)
plot(fit2)
plot(residuals(fit))
plot(residuals(fit2))
```

The plot shows that the random walk with drift model has much smaller forecast intervals. It has underestimated the forecast variance because of the heterogeneous residual.