Solutions to Exercises

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3.6 Exercises

Exercise 3.1

For the ETS(A,N,N) model,

$$\begin{aligned} y_t &= \ell_{t-1} + \varepsilon_t \\ \ell_t &= \ell_{t-1} + \alpha\varepsilon_t \end{aligned}$$

and $\boldsymbol{x}_t = \ell_{t-1}$. Therefore $\boldsymbol{w} = 1$, $\boldsymbol{F} = 1$, $\boldsymbol{g} = \alpha$ and $\boldsymbol{D} = \boldsymbol{F} - \boldsymbol{g}\boldsymbol{w}' = 1 - \alpha$.

Forecastability

$$\begin{aligned} c_j &= \boldsymbol{w}'\boldsymbol{D}^{j-1}\boldsymbol{g} = \alpha(1-\alpha)^{j-1} \\ \text{and} \qquad a_t &= \boldsymbol{w}'\boldsymbol{D}^{t-1}\boldsymbol{x}_0 = (1-\alpha)^{t-1}\ell_0. \end{aligned}$$

When $\alpha = 0$, $a_t = \ell_0$ and $c_j = 0$. Therefore $\sum_{j=1}^{\infty} |c_j| = 0$ and $\lim_{t\to\infty} a_t = \ell_0$, and so the process is forecastable.

When $\alpha = 2$, $a_t = (-1)^{t-1}\ell_0$ which does not converge as $t \to \infty$ and so the process is not forecastable.

Stationarity

$$\begin{aligned} d_t &= \boldsymbol{w}'\boldsymbol{F}^{t-1}\boldsymbol{x}_0 = \ell_0 \\ k_j &= \boldsymbol{w}'\boldsymbol{F}^{j-1}\boldsymbol{g} = \alpha, \qquad j \geq 1, \end{aligned}$$

and $k_0 = 1$. So for stationarity, we require $\sum_{j=0}^{\infty} |k_j| < \infty$.

When $\alpha = 0$, $k_j = 0$ for all $j \geq 1$ and so the process is stationary.

When $\alpha = 2$, $k_j = 2$ for all $j \geq 1$ and so the process is not stationary.

Exercise 3.2

[This question should have read "forecastable" rather than "stable".]

The local level with drift model is

$$\begin{aligned} y_t &= \ell_{t-1} + b + \varepsilon_t \\ \ell_t &= b + \ell_{t-1} + \alpha\varepsilon_t. \end{aligned}$$

Variable: $z_{1,t} = y_t - bt$: This can be written in state space form as

$$\begin{aligned} z_{1,t} &= \ell_{t-1} + b_{t-1} - u_{t-1} + \varepsilon_t \\ \ell_t &= b_{t-1} + \ell_{t-1} + \alpha\varepsilon_t \\ b_t &= b_{t-1} \\ u_t &= u_{t-1} + b_{t-1} \end{aligned}$$

where $b_0 = b$ and $u_0 = b$. Therefore, $\boldsymbol{x}_t = (\ell_t, b_t, u_t)'$, $\boldsymbol{w} = (1, 1, -1)'$, $\boldsymbol{g} = (\alpha, 0, 0)'$,

$$\boldsymbol{F} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \boldsymbol{D} = \boldsymbol{F} - \boldsymbol{g}\boldsymbol{w}' = \begin{bmatrix} 1-\alpha & 1-\alpha & \alpha \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\text{and} \quad \boldsymbol{D}^k = \begin{bmatrix} (1-\alpha)^k & (k-1) + (1-\alpha)^k & 1 - (1-\alpha)^{k+1}] \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}.$$

Therefore $\boldsymbol{w}'\boldsymbol{D}^k = \left[(1-\alpha)^k, (1-\alpha)^k, -(1-\alpha)^{k+1}\right]'$, $a_t = \boldsymbol{w}'\boldsymbol{D}^{t-1}\boldsymbol{x}_0 \to 0$, and $c_j = \boldsymbol{w}'\boldsymbol{D}^{j-1}\boldsymbol{g} = \alpha(1-\alpha)^{j-1}$. If $0 \le \alpha < 2$, then $\sum_{j=1}^{\infty} |c_j|$ is finite and so $z_{1,t}$ is forecastable.

Similarly, $d_t = \boldsymbol{w}'\boldsymbol{F}^{t-1}\boldsymbol{x}_0 = \ell_0$ and $k_j = \boldsymbol{w}'\boldsymbol{F}^{j-1}\boldsymbol{g} = \alpha$ for $j \ge 1$. So $\sum_{j=0}^{\infty} |k_j|$ is infinite and $z_{1,t}$ is not stationary.

Variable: $z_{2,t} = y_t - y_{t-1}$: This can be written in state space form as

$$\begin{aligned} z_{2,t} &= \ell_{t-1} - u_{t-1} + \varepsilon_t \\ \ell_t &= b_{t-1} + \ell_{t-1} + \alpha\varepsilon_t \\ b_t &= b_{t-1} \\ u_t &= \ell_{t-1} + \varepsilon_t \end{aligned}$$

where $b_0 = b$. Therefore, $\boldsymbol{x}_t = (\ell_t, b_t, u_t)'$, $\boldsymbol{w} = (1, 0, -1)'$, $\boldsymbol{g} = (\alpha, 0, 1)'$,

$$\boldsymbol{F} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{D} = \boldsymbol{F} - \boldsymbol{g}\boldsymbol{w}' = \begin{bmatrix} 1-\alpha & 1 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus

$$\boldsymbol{D}^k = \begin{bmatrix} (1-\alpha)^k & [1-(1-\alpha)^k]/\alpha & 1-(1-\alpha)^k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore $\boldsymbol{w}'\boldsymbol{D}^k = \left[(1-\alpha)^k,\ [1-(1-\alpha)^k]/\alpha,\ -(1-\alpha)^k\right]'$, $a_t = \boldsymbol{w}'\boldsymbol{D}^{t-1}\boldsymbol{x}_0 \to b_0/\alpha$, and $c_j = \boldsymbol{w}'\boldsymbol{D}^{j-1}\boldsymbol{g} = -(1-\alpha)^j$. So for $0 < \alpha < 2$, $\sum_{j=1}^{\infty} |c_j|$ is finite and $z_{2,t}$ is forecastable.

Similarly, $d_t = \boldsymbol{w}'\boldsymbol{F}^{t-1}\boldsymbol{x}_0 = \ell_0 - u_0$ and $k_j = \boldsymbol{w}'\boldsymbol{F}^{j-1}\boldsymbol{g} = 0$ for $j \ge 2$. So $\sum_{j=0}^{\infty} |k_j|$ is finite and $z_{1,t}$ is stationary.

Exercise 3.3

$$\begin{aligned}
y_t &= \ell_{t-1} + \varepsilon_t \\
&= \ell_{t-2} + \alpha\varepsilon_{t-1} + \varepsilon_t \\
&\dots \\
&= \ell_0 + \varepsilon_t + \sum_{j=1}^{t-1} \alpha\varepsilon_{t-j}
\end{aligned}$$

Therefore

$$\mathrm{E}(y_t \mid \ell_0) = \ell_0$$

and

$$\mathrm{Var}(y_t \mid \ell_0) = \sigma^2 + (t-1)\alpha^2\sigma^2 = [1 + (t-1)\alpha^2]\sigma^2.$$

[Note that the book has a typo here and replaces σ^2 by ℓ_0^2.]

Exercise 3.4

ETS(A,$\mathrm{A_d}$,N)

$$\begin{aligned}
y_t &= \ell_{t-1} + \phi b_{t-1} + \varepsilon_t \\
\ell_t &= \ell_{t-1} + \phi b_{t-1} + \alpha\varepsilon_t \\
b_t &= \phi b_{t-1} + \beta\varepsilon_t
\end{aligned}$$

Therefore $\boldsymbol{x}_t = (\ell_{t-1}, b_{t-1})'$, $\boldsymbol{w}_t = (1, \phi)'$, $\boldsymbol{F} = \begin{bmatrix} 1 & \phi \\ 0 & \phi \end{bmatrix}$, $\boldsymbol{g} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, and

$$\boldsymbol{D} = \boldsymbol{F} - \boldsymbol{g}\boldsymbol{w}' = \begin{bmatrix} 1 & \phi \\ 0 & \phi \end{bmatrix} - \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \begin{bmatrix} 1 & \phi \end{bmatrix} = \begin{bmatrix} 1-\alpha & \phi(1-\alpha) \\ -\beta & \phi(1-\beta) \end{bmatrix}.$$

To find the eigenvalues of $\boldsymbol{D}$, we must solve:

$$\det(\boldsymbol{D} - \boldsymbol{I}\lambda) = \det \begin{bmatrix} 1-\alpha-\lambda & \phi(1-\alpha) \\ -\beta & \phi(1-\beta)-\lambda \end{bmatrix} = (1-\alpha-\lambda)(\phi(1-\beta)-\lambda) + \beta\phi(1-\alpha) = 0$$

Therefore

$$\lambda^2 - \lambda[\phi(1-\beta) + (1-\alpha)] + \phi(1-\alpha) = 0,$$

and

$$\lambda = \frac{(1-\alpha) + \phi(1-\beta) \pm \sqrt{[(1-\alpha) + \phi(1-\beta)]^2 - 4\phi(1-\alpha)}}{2}.$$