Solutions to Exercises

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2.9 Exercises

Exercise 2.1

a. ETS(A,$\mathrm{A_d}$,N)

$$\begin{aligned} y_t &= \ell_{t-1} + \phi b_{t-1} + \varepsilon_t \\ \ell_t &= \ell_{t-1} + \phi b_{t-1} + \alpha\varepsilon_t \\ b_t &= \phi b_{t-1} + \beta\varepsilon_t \end{aligned}$$

b.

$$\begin{aligned} \boldsymbol{x}_t &= \begin{bmatrix} \ell_t & b_t \end{bmatrix}' \\ y_t &= \begin{bmatrix} 1 & \phi \end{bmatrix} \boldsymbol{x}_{t-1} + \varepsilon_t \\ \boldsymbol{x}_t &= \begin{bmatrix} 1 & \phi \\ 0 & \phi \end{bmatrix} \boldsymbol{x}_{t-1} + \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \varepsilon_t \end{aligned}$$

$$\begin{aligned} \boldsymbol{w}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix} 1 & \phi \end{bmatrix} \boldsymbol{x}_{t-1} & \boldsymbol{r}(\boldsymbol{x}_{t-1}) &= 1 \\ \boldsymbol{f}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix} 1 & \phi \\ 0 & \phi \end{bmatrix} \boldsymbol{x}_{t-1} & \boldsymbol{g}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \end{aligned}$$

c. ETS(A,A,A)

$$\begin{aligned} y_t &= \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t \\ \ell_t &= \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t \\ b_t &= b_{t-1} + \beta\varepsilon_t \\ s_t &= s_{t-m} + \gamma\varepsilon_t \end{aligned}$$

$$\begin{aligned} \boldsymbol{x}_t &= \begin{bmatrix} \ell_t & b_t & s_t \end{bmatrix}' \\ y_t &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \boldsymbol{x}_{t-1} + \varepsilon_t \\ \boldsymbol{x}_t &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x}_{t-1} + \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \varepsilon_t \end{aligned}$$

$$\begin{aligned} \boldsymbol{w}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \boldsymbol{x}_{t-1} & \boldsymbol{r}(\boldsymbol{x}_{t-1}) &= 1 \\ \boldsymbol{f}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x}_{t-1} & \boldsymbol{g}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \end{aligned}$$

d. ETS(M,A_d,N)

$$\begin{aligned}
y_t &= (\ell_{t-1} + \phi b_{t-1})(1 + \varepsilon_t) \\
\ell_t &= (\ell_{t-1} + \phi b_{t-1})(1 + \alpha\varepsilon_t) \\
b_t &= \phi b_{t-1} + \beta(\ell_{t-1} + \phi b_{t-1})\varepsilon_t
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{x}_t &= \begin{bmatrix}\ell_t & b_t\end{bmatrix}' \\
y_t &= \begin{bmatrix}1 & \phi\end{bmatrix} \boldsymbol{x}_{t-1}(1 + \varepsilon_t) \\
\boldsymbol{x}_t &= \begin{bmatrix}1 & \phi \\ 0 & \phi\end{bmatrix} \boldsymbol{x}_{t-1} + \begin{bmatrix}1 & 1\end{bmatrix} \boldsymbol{x}_{t-1} \begin{bmatrix}\alpha \\ \beta\end{bmatrix} \varepsilon_t
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{w}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix}1 & \phi\end{bmatrix} \boldsymbol{x}_{t-1} & \boldsymbol{r}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix}1 & \phi\end{bmatrix} \boldsymbol{x}_{t-1} \\
\boldsymbol{f}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix}1 & \phi \\ 0 & \phi\end{bmatrix} \boldsymbol{x}_{t-1} & \boldsymbol{g}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix}1 & \phi\end{bmatrix} \boldsymbol{x}_{t-1} \begin{bmatrix}\alpha \\ \beta\end{bmatrix}
\end{aligned}$$

e. ETS(M,A_d,A)

$$\begin{aligned}
y_t &= (\ell_{t-1} + \phi b_{t-1} + s_{t-m})(1 + \varepsilon_t) \\
\ell_t &= \ell_{t-1} + \phi b_{t-1} + \alpha(\ell_{t-1} + \phi b_{t-1} + s_{t-m})\varepsilon_t \\
b_t &= \phi b_{t-1} + \beta(\ell_{t-1} + \phi b_{t-1} + s_{t-m})\varepsilon_t \\
s_t &= s_{t-m} + \gamma(\ell_{t-1} + \phi b_{t-1} + s_{t-m})\varepsilon_t
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{x}_t &= \begin{bmatrix}\ell_t & b_t & s_t & s_{t-1} & \dots & s_{t-m+2} & s_{t-m+1}\end{bmatrix}' \\
y_t &= \begin{bmatrix}1 & \phi & 0 & 0 & \dots & 0 & 1\end{bmatrix} \boldsymbol{x}_{t-1}(1 + \varepsilon_t) \\
\boldsymbol{x}_t &= \begin{bmatrix}1 & \phi & 0 & 0 & \dots & 0 & 0 \\ 0 & \phi & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0\end{bmatrix} \boldsymbol{x}_{t-1} + \begin{bmatrix}1 & \phi & 0 & 0 & \dots & 0 & 1\end{bmatrix} \boldsymbol{x}_{t-1} \begin{bmatrix}\alpha \\ \beta \\ \gamma \\ 0 \\ 0 \\ \vdots \\ 0\end{bmatrix} \varepsilon_t
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{w}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix}1 & \phi & 0 & 0 & \dots & 0 & 1\end{bmatrix} \boldsymbol{x}_{t-1} & \boldsymbol{r}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix}1 & \phi & 0 & 0 & \dots & 0 & 1\end{bmatrix} \boldsymbol{x}_{t-1} \\
\boldsymbol{f}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix}1 & \phi & 0 & 0 & \dots & 0 & 0 \\ 0 & \phi & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0\end{bmatrix} \boldsymbol{x}_{t-1} & \boldsymbol{g}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix}1 & \phi & 0 & 0 & \dots & 0 & 1\end{bmatrix} \boldsymbol{x}_{t-1} \begin{bmatrix}\alpha \\ \beta \\ \gamma \\ 0 \\ 0 \\ \vdots \\ 0\end{bmatrix}
\end{aligned}$$

Exercise 2.2

a. ETS(A,N,N)

$$
\begin{aligned}
y_t &= \ell_{t-1} + \varepsilon_t \\
\ell_t &= \ell_{t-1} + \alpha\varepsilon_t
\end{aligned}
$$

$$
\begin{aligned}
\hat{y}_{t+1|t} &= \mathrm{E}[y_{t+1} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_t + \varepsilon_{t+1} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_t \mid \boldsymbol{x}_t] = \ell_t \\
\hat{y}_{t+2|t} &= \mathrm{E}[y_{t+2} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_{t+1} + \varepsilon_{t+2} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_{t+1} \mid \boldsymbol{x}_t] = \ell_t \\
&\dots \\
\hat{y}_{t+h|t} &= \mathrm{E}[y_{t+h} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_{t+h-1} + \varepsilon_{t+h} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_{t+h-1} \mid \boldsymbol{x}_t] = \ell_t
\end{aligned}
$$

$$
\begin{aligned}
v_{t+1|t} &= \mathrm{var}(y_{t+1} \mid \boldsymbol{x}_t) = \mathrm{var}(\ell_t + \varepsilon_{t+1} \mid \boldsymbol{x}_t) = \mathrm{var}(\varepsilon_{t+1} \mid \boldsymbol{x}_t) = \sigma^2 \\
v_{t+2|t} &= \mathrm{var}(y_{t+2} \mid \boldsymbol{x}_t) = \mathrm{var}(\ell_{t+1} + \varepsilon_{t+2} \mid \boldsymbol{x}_t) = \mathrm{var}(\ell_t + \alpha\varepsilon_{t+1} + \varepsilon_{t+2} \mid \boldsymbol{x}_t) \\
&= (1+\alpha^2)\sigma^2 \\
v_{t+3|t} &= \mathrm{var}(y_{t+3} \mid \boldsymbol{x}_t) = \mathrm{var}(\ell_{t+2} + \varepsilon_{t+3} \mid \boldsymbol{x}_t) = \mathrm{var}(\ell_t + \alpha\varepsilon_{t+1} + \alpha\varepsilon_{t+2} + \varepsilon_{t+3} \mid \boldsymbol{x}_t) \\
\\
&= \alpha^2\sigma^2 + \alpha^2\sigma^2 + \sigma^2 = (1+2\alpha^2)\sigma^2 \\
&\dots \\
v_{t+h|t} &= \mathrm{var}(y_{t+h} \mid \boldsymbol{x}_t) = \left[1 + (h-1)\alpha^2\right]\sigma^2
\end{aligned}
$$

b. ETS(A,A,N)

$$
\begin{aligned}
y_t &= \ell_{t-1} + b_{t-1}\varepsilon_t \\
\ell_t &= \ell_{t-1} + b_{t-1}\alpha\varepsilon_t \\
b_t &= b_{t-1} + \beta\varepsilon_t
\end{aligned}
$$

$$
\begin{aligned}
\hat{y}_{t+1|t} &= \mathrm{E}[y_{t+1} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_t + b_t + \varepsilon_{t+1} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_t + b_t \mid \boldsymbol{x}_t] = \ell_t + b_t \\
\hat{y}_{t+2|t} &= \mathrm{E}[y_{t+2} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_{t+1} + b_{t+1}\varepsilon_{t+2} \mid \boldsymbol{x}_t] = \ell_t + b_t + b_t = \ell_t + 2b_t \\
&\dots \\
\hat{y}_{t+h|t} &= \mathrm{E}[y_{t+h} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_{t+h-1} + b_{t+h-1} + \varepsilon_{t+h} \mid \boldsymbol{x}_t] = \ell_t + hb_t
\end{aligned}
$$

$$
\begin{aligned}
v_{t+1|t} &= \mathrm{var}(y_{t+1} \mid \boldsymbol{x}_t) = \mathrm{var}(\ell_t + b_t + \varepsilon_{t+1} \mid \boldsymbol{x}_t) = \sigma^2 \\
v_{t+2|t} &= \mathrm{var}(y_{t+2} \mid \boldsymbol{x}_t) = \mathrm{var}(\ell_{t+1} + b_{t+1} + \varepsilon_{t+2} \mid \boldsymbol{x}_t) = \mathrm{var}(\ell_t + b_t + \alpha\varepsilon_{t+1} + b_t + \beta\varepsilon_{t+1} + \varepsilon_{t+2} \mid \boldsymbol{x}_t) \\
&= (\alpha+\beta)^2\sigma^2 + \sigma^2 = [1 + ((\alpha+\beta)^2]\sigma^2 \\
v_{t+3|t} &= \mathrm{var}(y_{t+3} \mid \boldsymbol{x}_t) = \mathrm{var}(\ell_{t+2} + b_{t+2} + \varepsilon_{t+3} \mid \boldsymbol{x}_t) \\
&= \mathrm{var}(\ell_{t+1} + b_{t+1} + \alpha\varepsilon_{t+2} + b_{t+1} + \beta\varepsilon_{t+2} + \varepsilon_{t+3} \\
&= \mathrm{var}[\ell_t + b_t + \alpha\varepsilon_{t+1} + 2(b_t + \beta\varepsilon_{t+1}) + \alpha\varepsilon_{t+2} + \beta\varepsilon_{t+2} + \varepsilon_{t+3}] \\
&= \mathrm{var}[(\alpha+2\beta)\varepsilon_{t+1} + (\alpha+\beta)\varepsilon_{t+2} + \varepsilon_{t+3}] \\
&= [1 + (\alpha+\beta)^2 + (\alpha+2\beta)^2]\sigma^2 = \left[1 + \sum_{j=1}^{2}(\alpha + j\beta^2)\right]\sigma^2 \\
&\dots \\
v_{t+h|t} &= \left[1 + \sum_{j=1}^{h-1}(\alpha + j\beta^2)\right]\sigma^2
\end{aligned}
$$

c. ETS(M,N,N)

$$y_t = \ell_{t-1}(1+\varepsilon_t)$$
$$\ell_t = \ell_{t-1}(1+\alpha\varepsilon_t)$$

$$\begin{aligned}
\hat{y}_{t+1/h} &= \mathrm{E}[y_{t+1} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_t(1+\varepsilon_{t+1}) \mid \boldsymbol{x}_t] = \ell_t \\
\hat{y}_{t+2/h} &= \mathrm{E}[y_{t+2} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_{t+1}(1+\varepsilon_{t+2}) \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_t(1+\varepsilon_{t+1})(1+\varepsilon_{t+2})] = \ell_t \\
&\dots \\
\hat{y}_{t+h/h} &= \mathrm{E}[y_{t+h} \mid \boldsymbol{x}_t] = \ell_t
\end{aligned}$$

$$\begin{aligned}
v_{t+1|t} &= \mathrm{var}(y_{t+1} \mid \boldsymbol{x}_t) = \mathrm{var}(\ell_t(1+\varepsilon_{t+1}) \mid \boldsymbol{x}_t) = \ell_t^2\sigma^2 \\
v_{t+2|t} &= \mathrm{var}(y_{t+2} \mid \boldsymbol{x}_t) = \mathrm{var}(\ell_{t+1}(1+\varepsilon_{t+2}) \mid \boldsymbol{x}_t) = \mathrm{var}[\ell_t(1+\alpha\varepsilon_{t+1})(1+\varepsilon_{t+2}) \mid \boldsymbol{x}_t] \\
&= \ell_t^2 \mathrm{var}[(1+\alpha\varepsilon_{t+1})(1+\varepsilon_{t+2})] \\
&= \ell_t^2 \mathrm{var}[1+\alpha\varepsilon_{t+1}+\varepsilon_{t+2}+\alpha\varepsilon_{t+1}\varepsilon_{t+2}] \\
&= \ell_t^2[\alpha^2\mathrm{var}(\varepsilon_{t+1}) + \mathrm{var}(\varepsilon_{t+2}) + \alpha^2\mathrm{var}(\varepsilon_{t+1}\varepsilon_{t+2}) + 2\alpha^2\mathrm{cov}(\varepsilon_{t+1},\varepsilon_{t+1}\varepsilon_{t+2}) + 2\alpha\mathrm{cov}(\varepsilon_{t+2},\varepsilon_{t+1}\varepsilon_{t+2})] \\
&= \ell_t^2[\alpha^2\sigma^2 + \sigma^2 + \alpha^2\sigma^2\sigma^2] \\
&= \ell_t^2[(1+\alpha^2\sigma^2)(1+\sigma^2) - 1]
\end{aligned}$$

Exercise 2.3

```
> (bonds.ets <- ets(bonds))
ETS(A,Ad,N)

Call:
 ets(y = bonds)

  Smoothing parameters:
    alpha = 0.9999
    beta  = 0.1608
    phi   = 0.8

  Initial states:
    l = 5.5163
    b = 0.2967

  sigma:  0.2394

     AIC     AICc      BIC
256.1641 256.6683 270.3056

> (usnet.ets <- ets(usnetelec))
ETS(M,Md,N)

Call:
 ets(y = usnetelec)

  Smoothing parameters:
    alpha = 0.9999
    beta  = 1e-04
    phi   = 0.9638

  Initial states:
    l = 262.6421
    b = 1.1238
```

```
  sigma:  0.0236

     AIC     AICc      BIC
628.1943 629.4188 638.2310

> (ukc.ets <- ets(ukcars))
ETS(A,N,A)

Call:
 ets(y = ukcars)

  Smoothing parameters:
    alpha = 0.6267
    gamma = 2e-04

  Initial states:
    l = 338.4757
    s=-0.5313 -45.3246 20.6084 25.2476

  sigma:  25.3264

     AIC     AICc      BIC
1276.592 1277.385 1292.957

> (visit.ets <- ets(visitors))
ETS(M,A,M)

Call:
 ets(y = visitors)

  Smoothing parameters:
    alpha = 0.6244
    beta  = 1e-04
    gamma = 0.1832

  Initial states:
    l = 86.3534
    b = 2.0306
    s=0.942 1.076 1.0515 0.9568 1.3621 1.1157
           1.011 0.8294 0.9336 1.0017 0.8649 0.8554

  sigma:  0.0515

     AIC     AICc      BIC
2598.193 2600.632 2653.883
```

Exercise 2.4

```
> forecast(bonds.ets,h=4,level=80)
         Point Forecast    Lo 80    Hi 80
Jun 2004       4.791887 4.485047 5.098727
Jul 2004       4.865425 4.364764 5.366085
Aug 2004       4.924255 4.266113 5.582396
Sep 2004       4.971319 4.172657 5.769980

> forecast(usnet.ets,h=4,level=80)
     Point Forecast    Lo 80    Hi 80
2004       3905.517 3789.206 4026.131
2005       3963.887 3793.446 4133.019
2006       4020.971 3804.828 4224.363
2007       4076.769 3829.443 4314.189

> forecast(ukc.ets,h=4,level=80)
        Point Forecast    Lo 80    Hi 80
2005 Q2       426.8056 394.3485 459.2626
2005 Q3       360.8705 322.5657 399.1753
2005 Q4       405.6569 362.2828 449.0310
2006 Q1       431.4437 383.5363 479.3510

> forecast(visit.ets,h=4,level=80)
         Point Forecast    Lo 80    Hi 80
May 2005       361.1182 337.3021 384.9342
Jun 2005       396.1179 365.3447 426.8912
Jul 2005       494.4950 451.0785 537.9114
Aug 2005       428.0406 386.6065 469.4748
```